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OPTIMAL CONTROL AND ITS APPLICATION TO THE LIFE-CYCLE SAVINGS PROBLEM

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Master of Science

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Table of Contents

Acknowledgements	iv
List of Figures	vii
List of Tables	viii
Abstract	ix
1 Introduction to Optimal Control and its Applications	1
2 Necessary Conditions	4
2.1 Pontryagin Maximum Principle	4
3 Life-Cycle Saving Under Uncertain Lifetime Model	10
3.1 Model Development and Optimal Solutions	10
3.2 Sensitivity Analysis	17
3.2.1 Sensitivity of Model to Initial Value of Wealth	19
3.2.2 Sensitivity of Model to Discount Factor α	22
3.3 Model Implementation	26
3.3.1 Sensitivity Analysis on the Relative Risk Aversion Coefficient	26
3.3.2 MATLAB Simulation	32
4 Conclusion	41

Bibliography

42

List of Figures

3.1	Each plot below displays the optimal consumption path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 3$	34
3.2	Each plot below displays the associated optimal accumulation of wealth path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 3$	35
3.3	Each plot below displays the optimal consumption path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 1$	36
3.4	Each plot below displays the associated optimal accumulation of wealth path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 1$	37
3.5	Each plot below displays the optimal consumption path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 0.5$	38
3.6	Each plot below displays the associated optimal accumulation of wealth path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 0.5$	39

List of Tables

3.1	A selection of parameters from Leung's "Uncertain Lifetime, The Theory of the Consumer, and the Life Cycle Hypothesis"	33
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Abstract

Throughout the course of this thesis, we give an introduction to optimal control theory and its necessary conditions, prove Pontryagin's Maximum Principle, and present the life-cycle saving under uncertain lifetime optimal control problem. We present a very involved sensitivity analysis that determines how a change in the initial wealth, discount factor, or relative risk aversion coefficient may affect the model the terminal depletion of wealth time, optimal consumption path, and optimal accumulation of wealth path. Through simulation of the life-cycle saving under uncertain lifetime model, we are not only able to present the model dynamics through time, but also to demonstrate the feasibility of the model.

Chapter 1

Introduction to Optimal Control and its Applications

When mathematically modeling a physical system, it is not uncommon to make an attempt to optimize some outcome by varying the parameters of that system. Thanks to the work of Lev Pontryagin and Richard Bellman in the 1950's, optimal control theory was born and quickly became a standard approach to finding a set of control strategies that will optimize the outcome of a physical system. Today, optimal control theory is a commonly used optimization technique - especially for the fields of engineering, physical sciences and economics.

The process of setting up an optimal control problem typically involves three main components:

1. A deterministic mathematical model that describes the evolution over time of the physical system that will be controlled,
2. A clear indicator of the performance of the model - e.g., Will you be minimizing or maximizing the outcome? In an economic model, you may wish to maximize profits or minimize costs,

3. A set of variable constraints that properly define the components of the model.

We will define the set of admissible controls as the set of control functions that abide by the constraints of the model. The optimal admissible controls will give the optimal performance of the model. Throughout the rest of the paper, optimal controls will be denoted with the asterisk symbol (*).

Let's now demonstrate an economic application of optimal control theory. Consider a small business that needs to produce 250 products to fill an order by time T . A reasonable business will strive to minimize the overall cost of filling the order. Consider also that the unit production costs will increase linearly with the rate of production and the cost to store each product will be held constant per unit time. Define $x(t)$ as the total number of products that have been produced by time t . By definition, $x(0) = 0$ and $x(T) = 250$. The time derivative of x will then represent the instantaneous rate of change of inventory, or the rate of production. The total cost C of production at time t will then be

$$C(t) = [c_1 \dot{x}(t)] \dot{x}(t) + c_2 x(t), \quad (1.1)$$

where c_1 and c_2 are constants. Notice that the first term in equation (1.1), $[c_1 \dot{x}(t)] \dot{x}(t)$, represents the production costs and the second, $c_2 x(t)$ represents the cost of holding inventory. The optimal control problem then becomes

$$\min_{\dot{x}} \int_0^T c_1 (\dot{x}(t))^2 + c_2 x(t) dt \quad (1.2)$$

$$x(0) = 0 \quad (1.3)$$

$$x(T) = 250 \quad (1.4)$$

$$\dot{x}(t) \geq 0, \quad (1.5)$$

where we are trying to find an optimal rate of production $\dot{x}^*(t)$ and an optimal accumulation of inventory $x^*(t)$. To minimize the total cost of production for $t \in [0, T]$, we

may use optimal control techniques to manipulate our control variable $\dot{x}(t)$. Notice that the rate of production $\dot{x}(t)$ need not be continuous everywhere, but rather piecewise continuous.

Definition 1. A function $f : [a, b] \rightarrow \mathbb{R}$ is *piecewise continuous* on an interval $[a, b]$ if and only if

1. There exists a partition on the interval $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$ and x_1, x_2, \dots, x_{n-1} are discontinuities in the graph,
2. $f(x)$ is continuous on (x_i, x_{i+1}) for $i = 0, 1, \dots, n-1$,
3. $\lim_{x \rightarrow x_i^-} f(x) \neq +\infty, -\infty$, and
4. $\lim_{x \rightarrow x_i^+} f(x) \neq +\infty, -\infty$.

We explore a much more involved and in depth economic application to optimal control theory in Chapter 3, which relies heavily on the work completed by Siu Fai Leung [3–5].

Chapter 2

Necessary Conditions

2.1 Pontryagin Maximum Principle

Theorem 2.1.1 (Pontryagin's Maximum Principle). *Suppose that $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuously differentiable. Let $\mathcal{C}[t_0, t_1]$ be the set of all continuous functions with domain $[t_0, t_1]$. Now consider the optimization problem*

$$\max_{\mathbf{u} \in \mathbb{U}} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t)) dt, \quad (2.1)$$

subject to the following constraints:

$$\begin{cases} \dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)) & i = 1, \dots, n \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \mathbb{U} = \{\mathbf{u} : [t_0, t_1] \rightarrow \mathbb{R}^k, \mathbf{u} \in \mathcal{C}([t_0, t_1])\} \end{cases} \quad (2.2)$$

where \mathbb{U} is the set of all admissible controls, is non empty, and is open. Define \mathbf{u}^ as the set of optimal admissible controls and \mathbf{x}^* the associated optimal trajectory. Then there exists a continuous*

Lagrange multiplier $\lambda^* : [t_0, t_1] \rightarrow \mathbb{R}^n$ such that

$$\nabla_{\mathbf{u}} H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t)) = \mathbf{0}, \quad \forall t \in [t_0, t_1], \quad (2.3)$$

$$\nabla_{\mathbf{x}} H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t)) = -\dot{\lambda}^*(t), \quad \forall t \in [t_0, t_1], \quad (2.4)$$

$$\lambda^*(t_1) = \mathbf{0}, \quad (2.5)$$

where the Hamiltonian function H is defined to be

$$H(t, \mathbf{x}, \mathbf{u}, \lambda) = f(t, \mathbf{x}, \mathbf{u}) + \lambda \cdot g(t, \mathbf{x}, \mathbf{u}) \quad (2.6)$$

[1].

We will prove in detail the Pontryagin Maximum Principle for $n = 3$ under the regularity assumptions stated in Theorem 2.1.1. Our proof relies on the following two lemmas:

Lemma 2.1.2. Suppose $f(x) \in \mathcal{C}[a, b]$ and $\int_a^b f(x) \cdot g(x) dx = 0 \ \forall g(x) \in \mathcal{C}[a, b]$. Then $f(x) = 0$ on the entire interval $[a, b]$.

Lemma 2.1.3. (Leibniz). Suppose $F : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is continuous and $\nabla_{\mathbf{h}} F(t, \mathbf{h})$ exists and is continuous in (t, \mathbf{h}) . Then $\int_a^b F(t, \mathbf{h}) dt$ is differentiable and $\frac{d}{d\mathbf{h}} \int_a^b F(t, \mathbf{h}) dt = \int_a^b \frac{\partial}{\partial \mathbf{h}} F(t, \mathbf{h}) dt$ [8].

The proof of the Pontryagin Maximum Principle for $n = 3$ is as follows.

Proof. Let $J(\mathbf{u}) = \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t)) dt$. Define $\mathbf{u}^* \in \mathcal{C}$ with $k = 3$ to be an optimal control and \mathbf{x}^* to be the associated trajectory. Let us fix a continuous function $\mathbf{h} = (h_1, h_2, h_3) : [t_0, t_1] \rightarrow \mathbb{R}^3$. For every displacement constant $\epsilon \in \mathbb{R}^3$ we define the function $\mathbf{u}_\epsilon : [t_0, t_1] \rightarrow \mathbb{R}^3$ as

$$\mathbf{u}_\epsilon = \mathbf{u}^* + \epsilon \cdot \mathbf{h} = (u_1^* + \epsilon_1 h_1, u_2^* + \epsilon_2 h_2, u_3^* + \epsilon_3 h_3). \quad (2.7)$$

Since \mathbb{U} is a nonempty and open set of continuous admissible controls, we can say that there exists an $\boldsymbol{\epsilon}$ with $\|\boldsymbol{\epsilon}\|$ sufficiently small, $\mathbf{u}_{\boldsymbol{\epsilon}}$ is an admissible control. We denote $\mathbf{x}_{\boldsymbol{\epsilon}} : [t_0, t_1] \rightarrow \mathbb{R}^3$ to be the associated trajectory for $\mathbf{u}_{\boldsymbol{\epsilon}}$. Then we can define a function $\mathcal{J}_{\mathbf{h}} : \mathbb{R}^3 \rightarrow \mathbb{R}$ as

$$\mathcal{J}_{\mathbf{h}}(\boldsymbol{\epsilon}) = \int_{t_0}^{t_1} f(t, \mathbf{x}_{\boldsymbol{\epsilon}}(t), \mathbf{u}_{\boldsymbol{\epsilon}}(t)) dt \quad (2.8)$$

It is clear that when $\boldsymbol{\epsilon} = \mathbf{0}$, $\mathbf{u}_0(t) = \mathbf{u}^*(t)$, $\mathbf{x}_0(t) = \mathbf{x}^*(t)$, and $\mathbf{x}_{\boldsymbol{\epsilon}}(t_0) = \boldsymbol{\alpha}$. Then

$$\mathcal{J}_{\mathbf{h}}(\mathbf{0}) = \int_{t_0}^{t_1} f(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) dt. \quad (2.9)$$

Therefore, since the control set, \mathbb{U} , is convex and \mathbf{u}^* is optimal, $\mathcal{J}_{\mathbf{h}}(\mathbf{0}) \geq \mathcal{J}_{\mathbf{h}}(\boldsymbol{\epsilon})$ for all $\boldsymbol{\epsilon}$. Then $\mathcal{J}_{\mathbf{h}}$ has a local maximum at $\boldsymbol{\epsilon} = \mathbf{0}$. Thus, $\nabla_{\boldsymbol{\epsilon}} \mathcal{J}_{\mathbf{h}}(\mathbf{0}) = \mathbf{0}$. Let $\boldsymbol{\lambda} : [t_0, t_1] \rightarrow \mathbb{R}^3$ be a continuous function. Recall the constraint, $\dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \mathbf{u}(t))$, from our original optimization problem defined in the statement of the theorem. Then we have

$$\begin{aligned} \mathcal{J}_{\mathbf{h}}(\boldsymbol{\epsilon}) &= \int_{t_0}^{t_1} f(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}) dt \\ &= \int_{t_0}^{t_1} [f(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}) + \boldsymbol{\lambda}(g(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}) - \dot{\mathbf{x}}_{\boldsymbol{\epsilon}})] dt \\ &= \int_{t_0}^{t_1} [H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) - \boldsymbol{\lambda} \dot{\mathbf{x}}_{\boldsymbol{\epsilon}}] dt \\ &= \int_{t_0}^{t_1} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) dt - \int_{t_0}^{t_1} \boldsymbol{\lambda} \dot{\mathbf{x}}_{\boldsymbol{\epsilon}} dt. \end{aligned} \quad (2.10)$$

We can integrate the right most integral by using integration by parts:

$$\begin{aligned} \mathcal{J}_{\mathbf{h}}(\boldsymbol{\epsilon}) &= \int_{t_0}^{t_1} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) dt - [(\boldsymbol{\lambda} \mathbf{x}_{\boldsymbol{\epsilon}})|_{t_0}^{t_1} - \int_{t_0}^{t_1} \mathbf{x}_{\boldsymbol{\epsilon}} \dot{\boldsymbol{\lambda}} dt] \\ &= \int_{t_0}^{t_1} [H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) + \mathbf{x}_{\boldsymbol{\epsilon}} \dot{\boldsymbol{\lambda}}] dt - (\boldsymbol{\lambda} \mathbf{x}_{\boldsymbol{\epsilon}})|_{t_0}^{t_1} \\ &= \int_{t_0}^{t_1} [H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) + \mathbf{x}_{\boldsymbol{\epsilon}} \dot{\boldsymbol{\lambda}}] dt - \boldsymbol{\lambda}(t_1) \mathbf{x}_{\boldsymbol{\epsilon}}(t_1) + \boldsymbol{\lambda}(t_0) \mathbf{x}_{\boldsymbol{\epsilon}}(t_0). \end{aligned} \quad (2.11)$$

Then by taking the partial derivative of $\mathcal{J}_{\mathbf{h}}(\boldsymbol{\epsilon})$ with respect to ϵ_i for $1 \leq i \leq 3$, we have

$$\begin{aligned} \frac{\partial \mathcal{J}_{\mathbf{h}}(\boldsymbol{\epsilon})}{\partial \epsilon_i} = & \int_{t_0}^{t_1} [\nabla_{\mathbf{x}} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t) + \nabla_{\mathbf{u}} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{u}_{\boldsymbol{\epsilon}}(t) + \\ & + \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}} \dot{\boldsymbol{\lambda}}] dt - \boldsymbol{\lambda}(t_1) \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1) + \boldsymbol{\lambda}(t_0) \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_0). \end{aligned} \quad (2.12)$$

Since we know that $\mathbf{u}_{\boldsymbol{\epsilon}} = (\mathbf{u}_1^* + \epsilon_1 \mathbf{h}_1, \mathbf{u}_2^* + \epsilon_2 \mathbf{h}_2, \mathbf{u}_3^* + \epsilon_3 \mathbf{h}_3)$, we can see that

$$\nabla_{\epsilon_1} \mathbf{u}_{\boldsymbol{\epsilon}} = (\mathbf{h}_1, 0, 0)$$

$$\nabla_{\epsilon_2} \mathbf{u}_{\boldsymbol{\epsilon}} = (0, \mathbf{h}_2, 0)$$

$$\nabla_{\epsilon_3} \mathbf{u}_{\boldsymbol{\epsilon}} = (0, 0, \mathbf{h}_3).$$

Then we have

$$\begin{aligned} \frac{\partial \mathcal{J}_{\mathbf{h}}(\boldsymbol{\epsilon})}{\partial \epsilon_i} = & \int_{t_0}^{t_1} [\nabla_{\mathbf{x}} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t) + \frac{\partial}{\partial \mathbf{u}_i} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \mathbf{h}_i(t) + \\ & + \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}} \dot{\boldsymbol{\lambda}}] dt - \boldsymbol{\lambda}(t_1) \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1) + \boldsymbol{\lambda}(t_0) \nabla_{\epsilon_i} \boldsymbol{\alpha} \\ = & \int_{t_0}^{t_1} [\nabla_{\mathbf{x}} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t) + \frac{\partial}{\partial \mathbf{u}_i} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \mathbf{h}_i(t) + \\ & + \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}} \dot{\boldsymbol{\lambda}}] dt - \boldsymbol{\lambda}(t_1) \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1). \end{aligned} \quad (2.13)$$

At $\boldsymbol{\epsilon} = \mathbf{0}$, this is

$$\begin{aligned} \frac{\partial \mathcal{J}_{\mathbf{h}}(\mathbf{0})}{\partial \epsilon_i} = & \int_{t_0}^{t_1} [\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) \cdot (\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t)|_{\boldsymbol{\epsilon}=\mathbf{0}}) + \frac{\partial}{\partial \mathbf{u}_i} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) \cdot \mathbf{h}_i(t) + \\ & + \dot{\boldsymbol{\lambda}} (\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=\mathbf{0}})] dt - \boldsymbol{\lambda}(t_1) (\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1)|_{\boldsymbol{\epsilon}=\mathbf{0}}). \end{aligned} \quad (2.14)$$

Define $\boldsymbol{\lambda}$ so that $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda})$ for $t \in [t_0, t_1]$ and $\boldsymbol{\lambda}(t_1) = \mathbf{0}$. Since we know that

$$\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) = \nabla_{\mathbf{x}} f(t, \mathbf{x}^*, \mathbf{u}^*) + \boldsymbol{\lambda} \cdot \nabla_{\mathbf{x}} f(t, \mathbf{x}^*, \mathbf{u}^*), \quad (2.15)$$

we also know that $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda})$ is linear in $\boldsymbol{\lambda}$ and f has continuous first partial

derivatives. Therefore, a unique solution λ^* exists. Now let $\lambda = \lambda^*$. Then (2.14) becomes

$$\begin{aligned} \frac{\mathcal{J}_h(\mathbf{0})}{\partial \epsilon_i} &= \int_{t_0}^{t_1} \left[\left(\nabla_x H(t, \mathbf{x}^*, \mathbf{u}^*, \lambda^*) + \dot{\lambda}^* \right) \cdot (\nabla_{\epsilon_i} \mathbf{x}_\epsilon|_{\epsilon=0}) \right. \\ &\quad \left. + \left[\frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \lambda) h_i(t) \right] dt - \lambda^*(t_1) (\nabla_{\epsilon_i} \mathbf{x}_\epsilon(t_1)|_{\epsilon=0}) \right] \\ &= \int_{t_0}^{t_1} \frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \lambda^*) h_i(t) dt. \end{aligned} \quad (2.16)$$

Since \mathcal{J}_h attains a maximum at $\epsilon = \mathbf{0}$, $\nabla_{\epsilon} \mathcal{J}_h(\mathbf{0}) = \mathbf{0}$ is defined to be equal to $\mathbf{0}$, we have

$$\mathbf{0} = \int_{t_0}^{t_1} \frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \lambda^*) h_i(t) dt. \quad (2.17)$$

for $i \in [1, 3]$. Therefore, $\frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \lambda^*) = 0$, proving Theorem 2.1.1 for $n = k = 3$ [1]. \square

Since we will be optimizing a control problem with state constraints in Chapter 3, we will require the use of the following maximum principle for optimal control problems with state constraints:

Theorem 2.1.4. *Consider the following optimal control problem with state constraints:*

$$\max_{\mathbf{u} \in \mathbb{U}} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t)) dt \quad (2.18)$$

$$\dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(t_0) = \alpha, \quad (2.19)$$

$$\mathbf{u}(t) \geq 0, \quad (2.20)$$

$$\mathbf{x}(t) \geq 0, \quad (2.21)$$

where f, g are piecewise continuously differentiable. Let \mathbf{u}^* be an optimal control and \mathbf{x}^* the associated optimal trajectory over the interval $[t_0, t_1]$. Suppose that \mathbf{u}^* is right-continuous with left-hand limits and satisfies the constraints defined above. Define the Lagrangian function to be

the following:

$$L(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(t, \mathbf{x}(t), \mathbf{u}(t)) + \boldsymbol{\lambda} \cdot \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) + \mu_1 g(t, \mathbf{x}(t), \mathbf{u}(t)) + \mu_2 h(\mathbf{x}(t), t). \quad (2.22)$$

Then there exists a piecewise absolutely continuous function $\boldsymbol{\lambda} : [t_0, t_1] \rightarrow \mathbb{R}$ and piecewise continuous functions $\mu_1, \mu_2 : [t_0, t_1] \rightarrow \mathbb{R}$ such that

$$\nabla_{\mathbf{u}} L(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}) = \mathbf{0}, \quad (2.23)$$

$$\nabla_{\mathbf{x}} L(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}) = -\dot{\boldsymbol{\lambda}}^*, \quad (2.24)$$

$$\mu_1(t) \geq 0, \quad \mu_1(t) g^* = 0, \quad (2.25)$$

$$\mu_2(t) \geq 0, \quad \mu_2(t) h^* = 0 \quad (2.26)$$

[2].

Definition 2. A function f is said to be absolutely continuous on the interval $[a, b]$ if f is defined on $[a, b]$, f' exists almost everywhere, f is Lebesgue integrable on $[a, b]$, and $f(x) = f(a) + \int_a^x f'(t) dt$ for $x \in [a, b]$ [6].

Chapter 3

Life-Cycle Saving Under Uncertain Lifetime Model

3.1 Model Development and Optimal Solutions

Suppose that a consumer expects to have a lifetime of T years and is working to find an optimal consumption plan that will allow them to earn and distribute wealth through savings and consumption. Suppose that this consumer begins with no wealth and wishes to use up all accumulated wealth by the time of his or her death. The preferences of the consumer may be given by the following Fisher utility function:

$$V(c) = \int_0^T \alpha(t)g[c(t)]dt, \quad (3.1)$$

where $\alpha(t)$ is a discount function, $c(t)$ is a consumption plan, and $g(c)$ is the utility function that is associated with the rate of consumption throughout time. Before going any further, we will make the following assumptions about the attributes of the Fisher utility function:

- $c(t)$ is piecewise continuous on $[0, T]$

- $\alpha(t)$ is positive and continuously differentiable on $[0, T]$
- $g(c)$ is a concave, continuously differentiable function on $(0, \infty)$, where $g'(c) > 0$ and $g''(c) < 0$.

We can also define the consumer's net wealth at time t as the following:

$$S(t) = \int_0^t \left[\left(e^{\int_\tau^t j(x) dx} \right) (m(\tau) - c(\tau)) \right] d\tau, \quad (3.2)$$

where $j(\tau)$ is the interest rate that begins at time τ and $m(\tau)$ is the income function. Assuming that $j(\tau)$ is non-negative and continuous on $[0, T]$, and $m(\tau)$ is non-negative and piecewise continuously differentiable on $[0, T]$, then $S(t)$ will be a piecewise continuously differentiable function on $[0, T]$. $S(t)$ represents the net accumulation of wealth with an interest rate that is compounded continuously throughout time. In order to prevent having a negative amount of accumulated wealth at the time of death, we will define the wealth constraint

$$S(T) \geq 0 \quad (3.3)$$

[9].

Now suppose that the consumer has an uncertain lifetime with a maximum possible lifetime length of \bar{T} . Thus, we will define T as a random variable with a probability density function π on the interval $[0, \bar{T}]$. Then $\int_0^{\bar{T}} \pi(t) dt = 1$ and $\pi(t) > 0$ for $0 < t < \bar{T}$. The probability of a consumer surviving past time t can then be defined as

$$\Omega(t) = \int_t^{\bar{T}} \pi(\tau) d\tau = e^{-\int_0^t \pi_x(x) dx}, \quad 0 \leq t \leq \bar{T}, \quad (3.4)$$

where $\pi_t(t) = \pi(t)/\Omega(t)$ is the mortality hazard function. The expected value of the consumer's preferences, represented by the Fisher utility function, is simply

$$\begin{aligned}\bar{V}(c) = \mathbb{E}[V(c)] &= \int_0^{\bar{T}} \pi(\tau) \int_0^{\tau} \alpha(t)g[c(t)] dt d\tau \\ &= \int_0^{\bar{T}} \alpha(t)g[c(t)] \int_t^{\bar{T}} \pi(\tau) d\tau dt\end{aligned}\quad (3.5)$$

Recalling that $\Omega(t) = \int_t^{\bar{T}} \pi(\tau) d\tau$, we can rewrite the expected utility for consumption plan c in the following way:

$$\bar{V}(c) = \int_0^{\bar{T}} \Omega(t)\alpha(t)g[c(t)]dt. \quad (3.6)$$

Our goal is to maximize the utility function $\bar{V}(c)$ with uncertain lifetime. To do so, we must find an optimal consumption plan $c(t) \in \Phi$, where Φ is the set of admissible controls. The problem then becomes:

$$\max_{c(t) \in \Phi} \bar{V}(c) = \max_{c(t) \in \Phi} \int_0^{\bar{T}} \Omega(t)\alpha(t)g[c(t)]dt \quad (3.7)$$

such that

$$c(t) \geq 0 \quad (3.8)$$

$$S(t) \geq 0 \quad (3.9)$$

$$S'(t) = j(t)S(t) + m(t) - c(t) \quad (3.10)$$

$$S(0) = S_0 \quad (3.11)$$

$$S(\bar{T}) = 0 \quad (3.12)$$

[4].

Before we can solve for the optimal consumption plan c^* , we must begin by defining

the Hamiltonian function

$$H(c, S, t, \lambda) = \Omega(t)\alpha(t)g[c(t)] + \lambda(t)[j(t)S(t) + m(t) - c(t)] \quad (3.13)$$

and the Lagrangian Function

$$\begin{aligned} L(c, S, t, \lambda, \mu_1, \mu_2) = & \Omega(t)\alpha(t)g[c(t)] + \lambda(t)[j(t)S(t) + m(t) - c(t)] \\ & + \mu_1(t)c(t) + \mu_2(t)S(t). \end{aligned} \quad (3.14)$$

From the necessary conditions of optimality, we have

$$\frac{\partial L(c^*(t), S^*(t), t, \mu_1, \mu_2)}{\partial c(t)} = \Omega(t)\alpha(t)g'[c^*(t)] - \lambda(t) + \mu_1(t) = 0 \quad (3.15)$$

$$\frac{\partial L(c^*(t), S^*(t), t, \mu_1, \mu_2)}{\partial S(t)} = \lambda(t)j(t) + \mu_2(t) = -\lambda'(t) \quad (3.16)$$

$$\mu_1(t) \geq 0, \mu_1(t)c^*(t) = 0 \quad (3.17)$$

$$\mu_2(t) \geq 0, \mu_2(t)S^*(t) = 0. \quad (3.18)$$

Through the use of an integrating factor, we can solve for $\lambda(t)$ in equation (3.16) above. Define the integrating factor to be $e^{\int_0^t j(x)dx}$. Then the solution to (3.16) is obtained as follows:

$$\begin{aligned} \lambda'(t) + \lambda(t)j(t) &= -\mu_2(t) \\ e^{\int_0^t j(x)dx}\lambda'(t) + e^{\int_0^t j(x)dx}\lambda(t) &= -e^{\int_0^t j(x)dx}\mu_2(t) \\ \int_0^t \frac{d}{dw} \left[e^{\int_0^w j(x)dx}\lambda(w) \right] dw &= - \int_0^t e^{\int_0^w j(x)dx}\mu_2(w)dw \\ e^{\int_0^t j(x)dx}\lambda(t) - e^{\int_0^0 j(x)dx}\lambda(0) &= - \int_0^t e^{\int_0^w j(x)dx}\mu_2(w)dw \\ e^{\int_0^t j(x)dx}\lambda(t) &= \lambda(0) - \int_0^t e^{\int_0^w j(x)dx}\mu_2(w)dw. \end{aligned} \quad (3.19)$$

Solving equation (3.19) for $\lambda(t)$ gives us

$$\begin{aligned}\lambda(t) &= \lambda(0)e^{-\int_0^t j(x)dx} - e^{-\int_0^t j(x)dx} \int_0^t e^{\int_0^w j(x)dx} \mu_2(w)dw \\ \lambda(t) &= \lambda(0)e^{-\int_0^t j(x)dx} - \int_0^t e^{-\int_w^t j(x)dx} \mu_2(w)dw.\end{aligned}\tag{3.20}$$

Now that we have solved for $\lambda(t)$, we can substitute equation (3.20) into the necessary condition defined in equation (3.15):

$$\begin{aligned}\Omega(t)\alpha(t)g'[c^*(t)] - \lambda(t) + \mu_1(t) &= 0 \\ \Omega(t)\alpha(t)g'[c^*(t)] - \lambda(0)e^{-\int_0^t j(x)dx} + \int_0^t e^{-\int_w^t j(x)dx} \mu_2(w)dw + \mu_1(t) &= 0.\end{aligned}\tag{3.21}$$

Solving equation (3.21) for $\Omega(t)\alpha(t)g'[c^*(t)]$ then gives us

$$\Omega(t)\alpha(t)g'[c^*(t)] = \lambda(0)e^{-\int_0^t j(x)dx} - \int_0^t e^{-\int_w^t j(x)dx} \mu_2(w)dw - \mu_1(t).\tag{3.22}$$

We will use the following proposition to continue

Proposition 1. *Assume (3.8)-(3.12). If either $\lim_{c \rightarrow 0^+} g'(c) < \infty$ or $m(\bar{T}) > 0$, then there exists a $t^* \in [0, \bar{T})$ such that $t^* = \min\{t \in [0, T) : S^*(z) = 0 \text{ and } c^*(z) = m(z) \text{ for all } z \in [t, \bar{T}]\}$.*

For this problem, we will assume that after retirement, and individual will have a constant income from a retirement annuity. Due to this assumption, we see that $m(\bar{T}) > 0$, and we will apply Proposition 1. Let $t = t^*$. Then by Proposition 1, $c^*(t^*) = m(t^*)$ and (3.22) becomes

$$\Omega(t^*)\alpha(t^*)g'[m(t^*)] = \lambda(0)e^{-\int_0^{t^*} j(x)dx} - \int_0^{t^*} e^{-\int_w^{t^*} j(x)dx} \mu_2(w)dw.\tag{3.23}$$

By solving equation (3.23) above for $\lambda(0)$, we have

$$\begin{aligned}\lambda(0) &= e^{\int_0^{t^*} j(x) dx} \Omega(t^*) \alpha(t^*) g'[m(t^*)] + e^{\int_0^{t^*} j(x) dx} \int_0^{t^*} e^{-\int_w^{t^*} j(x) dx} \mu_2(w) dw \\ &= e^{\int_0^{t^*} j(x) dx} \Omega(t^*) \alpha(t^*) g'[m(t^*)] + \int_0^{t^*} e^{\int_0^w j(x) dx} \mu_2(w) dw.\end{aligned}\quad (3.24)$$

If we now substitute equation (3.24) in for $\lambda(0)$ in (3.22), for $t \in [0, t^*]$ we have

$$\begin{aligned}\Omega(t) \alpha(t) g'[c^*(t)] &= e^{-\int_0^t j(x) dx} \left[e^{\int_0^{t^*} j(x) dx} \Omega(t^*) \alpha(t^*) g'[m(t^*)] + \int_0^{t^*} e^{\int_0^w j(x) dx} \mu_2(w) dw \right] \\ &\quad - \int_0^t e^{-\int_w^t j(x) dx} \mu_2(w) dw - \mu_1(t) \\ &= e^{\int_t^{t^*} j(x) dx} \Omega(t^*) \alpha(t^*) g'[m(t^*)] + \int_0^{t^*} e^{-\int_w^t j(x) dx} \mu_2(w) dw \\ &\quad - \int_0^t e^{-\int_w^t j(x) dx} \mu_2(w) dw - \mu_1(t).\end{aligned}$$

Thus,

$$\Omega(t) \alpha(t) g'[c^*(t)] = e^{\int_t^{t^*} j(x) dx} \Omega(t^*) \alpha(t^*) g'[m(t^*)] + \int_t^{t^*} e^{-\int_w^t j(x) dx} \mu_2(w) dw - \mu_1(t) \quad (3.25)$$

We can now use (3.25) to solve for the optimal consumption path $c^*(t)$:

$$c^*(t) = \begin{cases} g'^{-1} \left(\frac{e^{\int_t^{t^*} j(x) dx} \Omega(t^*) \alpha(t^*) g'[m(t^*)] + \int_t^{t^*} e^{-\int_w^t j(x) dx} \mu_2(w) dw - \mu_1(t)}{\Omega(t) \alpha(t)} \right), & t \in [0, t^*] \\ m(t), & t \in [t^*, \bar{T}]. \end{cases} \quad (3.26)$$

Since the accumulation of wealth function, $S(t)$, is also a controllable function that impacts the overall utility function, we will work to solve for the optimal accumulated savings path $S^*(t)$ that is associated with the optimal consumption path $c^*(t)$. To do

this, we will begin by solving the ordinary differential equation from (3.11):

$$\begin{aligned}
S'(t) &= j(t)S(t) + m(t) - c(t) \\
S'(t) - j(t)S(t) &= m(t) - c(t) \\
e^{-\int_0^t j(x)dx} S'(t) - j(t)e^{-\int_0^t j(x)dx} S(t) &= e^{-\int_0^t j(x)dx} (m(t) - c(t)) \\
\int_0^t \frac{d}{dz} \left[e^{-\int_0^z j(x)dx} S(z) \right] dz &= \int_0^t e^{-\int_0^z j(x)dx} (m(z) - c(z)) dz \\
e^{-\int_0^t j(x)dx} S(t) - S_0 &= \int_0^t e^{-\int_0^z j(x)dx} (m(z) - c(z)) dz. \tag{3.27}
\end{aligned}$$

Through simplification, we can solve (3.27) for the wealth function $S(t)$:

$$S(t) = e^{\int_0^t j(x)dx} \left[S_0 + \int_0^t e^{-\int_0^z j(x)dx} (m(z) - c(z)) dz \right]. \tag{3.28}$$

Now let $t = t^*$, where t^* is the time at which all wealth of the consumer is depleted.

Then $S(t^*) = 0$ and

$$0 = e^{\int_0^{t^*} j(x)dx} \left[S_0 + \int_0^{t^*} e^{-\int_0^t j(x)dx} (m(t) - c(t)) dt \right] \tag{3.29}$$

Note that $e^{\int_0^{t^*} j(x)dx} \neq 0$, and by rearranging equation (3.29), we have the following significant equality:

$$\int_0^{t^*} e^{-\int_0^t j(x)dx} c(t) dt = S_0 + \int_0^{t^*} e^{-\int_0^t j(x)dx} m(t) dt. \tag{3.30}$$

The equality shown in (3.30) signifies that the total consumption until all wealth is depleted at time $t = t^*$ must be equal to the initial wealth plus the total income up until that particular point in time. We may also note that at time $t = t^*$, the initial wealth is simply

$$S_0 = \int_0^{t^*} e^{-\int_0^t j(x)dx} (c(t) - m(t)) dt. \tag{3.31}$$

The *optimal* accumulation of wealth function $S^*(t)$ is simply the accumulation of wealth function that is associated with the optimal consumption path $c^*(t)$:

$$S^*(t) = \begin{cases} e^{\int_0^t j(x) dx} \left[S_0 + \int_0^t e^{-\int_0^z j(x) dx} (m(z) - c^*(z)) dz \right], & t \in [0, t^*] \\ 0, & t \in [t^*, \bar{T}]. \end{cases} \quad (3.32)$$

From this, we can say that once the total accumulation of wealth has been depleted at t^* , the accumulation of wealth will remain zero for the remainder of the lifetime of the consumer. From equations (3.26) and (3.31), the optimal terminal wealth depletion time t^* can be found with the following equation for S_0 :

$$S_0 = \int_0^{t^*} e^{-\int_0^t j(x) dx} \left[(g')^{-1} \left(\frac{e^{\int_t^{t^*} j(x) dx} \Omega(t^*) \alpha(t^*) g'[m(t^*)] + \int_t^{t^*} e^{-\int_w^t j(x) dx} \mu_2(w) dw - \mu_1(t)}{\Omega(t) \alpha(t)} \right) - m(t) \right] dt \quad (3.33)$$

[5].

3.2 Sensitivity Analysis

To further expand on the dynamics of the life-cycle savings under uncertain lifetime model, we should investigate how a sudden increase or decrease in the initial wealth and discount parameters will affect the consumer's overall optimal wealth, consumption, and time until total wealth is depleted. To do this, we must first make the following simplifying assumptions that will allow us to properly and efficiently perform the

analysis:

$$\alpha(t) = \exp(-\alpha t), \quad \text{for } \alpha \geq 0 \quad (3.34)$$

$$j(t) = j \quad (3.35)$$

$$\mu_1 = \mu_2 = 0, \quad (3.36)$$

where α and j are real valued constants. Note that the simplifying assumptions made will not affect the results of the sensitivity analysis. With the assumptions, the optimal consumption, initial wealth, and optimal accumulated wealth functions on $t \in [0, t^*]$ are simplified to

$$\begin{aligned} c^*(t) &= (g')^{-1} \left(\frac{e^{\int_t^{t^*} j dx} \Omega(t^*) e^{-\alpha t^*} g'(m(t^*))}{\Omega(t) e^{-\alpha t}} \right) \\ &= (g')^{-1} \left(\frac{e^{(j-\alpha)t^*} \Omega(t^*) g'(m(t^*))}{\Omega(t) e^{(j-\alpha)t}} \right). \end{aligned} \quad (3.37)$$

$$\begin{aligned} S_0 &= \int_0^{t^*} e^{-\int_0^t j dx} \left[(g')^{-1} \left(\frac{e^{\int_t^{t^*} j dx} \Omega(t^*) e^{-\alpha t^*} g'(m(t^*))}{\Omega(t) e^{-\alpha t}} \right) - m(t) \right] dt \\ &= \int_0^{t^*} e^{-jt} \left[(g')^{-1} \left(\frac{e^{(j-\alpha)t^*} \Omega(t^*) g'(m(t^*))}{\Omega(t) e^{(j-\alpha)t}} \right) - m(t) \right] dt. \end{aligned} \quad (3.38)$$

$$\begin{aligned} S^*(t) &= e^{\int_0^t j dx} \left[S_0 + \int_0^t e^{-\int_0^z j dx} (m(z) - c^*(z)) dz \right] \\ &= e^{jt} \left[S_0 + \int_0^t e^{-jz} (m(z) - c^*(z)) dz \right]. \end{aligned} \quad (3.39)$$

We also define the following functions that will allow us to simplify the results of our calculations throughout the entire analysis:

$$\sigma(z, t) = \frac{e^{(j-\alpha)z} \Omega(z) g'(m(z))}{\Omega(t) e^{(j-\alpha)t}} \quad (3.40)$$

$$\Delta(t) = j - \alpha - \pi_t(t) + \frac{g''(m(t)) m'(t)}{g'(m(t))} \quad (3.41)$$

$$\Psi(t) = \int_0^t e^{-jz} \left[\frac{g'(c^*(z))}{g''(c^*(z))} \right] dz. \quad (3.42)$$

We will use the following theorem to perform a sensitivity analysis on the model:

Theorem 3.2.1. (Leibniz). Suppose $f(x, t)$ is a function where $\frac{\partial f}{\partial t}$ exists. Then

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} dt + f(b(t), t) \cdot \frac{\partial b(t)}{\partial t} - f(a(t), t) \cdot \frac{\partial a(t)}{\partial t}.$$

3.2.1 Sensitivity of Model to Initial Value of Wealth

Sensitivity of Depletion of Wealth Time t^*

First we work to determine how the time until the depletion of wealth, t^* , is affected by a change in the initial wealth, S_0 . To do this, we must take the partial derivative of the function S_0 (equation 3.38) with respect to S_0 . Recall that t^* is dependent upon the parameter S_0 . We obtain

$$1 = e^{-jt^*} \left[(g')^{-1} \left(\frac{e^{(j-\alpha)t^*} \Omega(t^*) g'(m(t^*))}{\Omega(t^*) e^{(j-\alpha)t^*}} \right) - m(t^*) \right] \frac{\partial t^*}{\partial S_0} + \int_0^{t^*} e^{-jt} \frac{\partial (g')^{-1}(\sigma(t^*, t))}{\partial S_0} dt. \quad (3.43)$$

Notice that $(g')^{-1} \left(\frac{e^{(j-\alpha)t^*} \Omega(t^*) g'(m(t^*))}{\Omega(t^*) e^{(j-\alpha)t^*}} \right)$ is exactly $c^*(t^*)$ and that $c^*(t^*) = m(t^*)$.

Thus we now have:

$$\begin{aligned} 1 &= \int_0^{t^*} e^{-jt} \frac{\partial (g')^{-1}(\sigma(t^*, t))}{\partial S_0} dt \\ &= \int_0^{t^*} \frac{e^{-jt}}{g''((g')^{-1}(\sigma(t^*, t)))} \left[\frac{(j - \alpha) e^{(j-\alpha)t^*} \Omega(t^*) g'(m(t^*))}{\Omega(t) e^{(j-\alpha)t}} + \right. \\ &\quad \left. + \frac{e^{(j-\alpha)t^*} (\Omega'(t^*) g'(m(t^*)) + \Omega(t^*) g''(m(t^*)) m'(t^*))}{\Omega(t) e^{(j-\alpha)t}} \right] \frac{\partial t^*}{\partial S_0} dt. \end{aligned} \quad (3.44)$$

Recall that $\Omega(t) = e^{-\int_0^t \pi_x(x) dx}$. Then $\Omega'(t) = -\pi_t(t)e^{-\int_0^t \pi_x(x) dx} = -\pi_t(t)\Omega(t)$. Equation (3.44) may then be rewritten and simplified in the following manner:

$$\begin{aligned}
1 &= \int_0^{t^*} \frac{e^{-jt}}{g''((g')^{-1}(\sigma(t^*, t)))} \left[\frac{(j - \alpha)e^{(j-\alpha)t^*} \Omega(t^*) g'(m(t^*))}{\Omega(t) e^{(j-\alpha)t}} + \right. \\
&\quad \left. + \frac{e^{(j-\alpha)t^*} (-\pi_{t^*}(t^*) \Omega(t^*) g'(m(t^*)) + \Omega(t^*) g''(m(t^*)) m'(t^*))}{\Omega(t) e^{(j-\alpha)t}} \right] \frac{\partial t^*}{\partial S_0} dt \\
&= \int_0^{t^*} \frac{e^{-jt} \left[j - \alpha - \pi_{t^*}(t^*) + \frac{g''(m(t^*)) m'(t^*)}{g'(m(t^*))} \right] e^{(j-\alpha)t^*} \Omega(t^*) g'(m(t^*))}{g''((g')^{-1}(\sigma(t^*, t))) \Omega(t) e^{(j-\alpha)t}} \cdot \frac{\partial t^*}{\partial S_0} dt \\
&= \int_0^{t^*} \frac{e^{-jt} \Delta(t^*) \sigma(t^*, t)}{g''((g')^{-1}(\sigma(t^*, t)))} \cdot \frac{\partial t^*}{\partial S_0} dt. \tag{3.45}
\end{aligned}$$

Since $c^*(t) = (g')^{-1}(\sigma(t^*, t))$, we rewrite and simplify equation (3.45) to obtain the following:

$$\begin{aligned}
1 &= \Delta(t^*) \frac{\partial t^*}{\partial S_0} \int_0^{t^*} e^{-jt} \frac{g'(c^*(t))}{g''(c^*(t))} dt \\
&= \Delta(t^*) \Psi(t^*) \frac{\partial t^*}{\partial S_0}. \tag{3.46}
\end{aligned}$$

Therefore, the way in which the time until depletion of wealth changes with respect to the initial wealth can be represented by

$$\frac{\partial t^*}{\partial S_0} = \frac{1}{\Delta(t^*) \Psi(t^*)}. \tag{3.47}$$

Sensitivity of Optimal Consumption Path $c^*(t)$

We will perform a similar derivation to find out how a change in the initial wealth affects the optimal consumption path. The derivation is shown below:

$$\begin{aligned}
 \frac{\partial c^*(t)}{\partial S_0} &= \frac{\partial (g')^{-1}(\sigma(t^*, t))}{\partial S_0} = \frac{\partial (g')^{-1}(\sigma(t^*, t))}{\partial t^*} \left[\frac{\partial t^*}{\partial S_0} \right] \\
 &= \frac{\Delta(t^*)\sigma(t^*, t)}{g''((g')^{-1}(\sigma(t^*, t)))} \cdot \frac{1}{\Delta(t^*)\Psi(t^*)} \\
 &= \frac{\sigma(t^*, t)}{g''((g')^{-1}(\sigma(t^*, t)))\Psi(t^*)} \\
 &= \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)}.
 \end{aligned} \tag{3.48}$$

Sensitivity of Optimal Wealth Path $S^*(t)$

To find how a change in the initial wealth affects the overall optimal wealth function associated with the optimal consumption path, we will take the partial derivative of $S^*(t)$ with respect to S_0 .

$$\begin{aligned}
 \frac{\partial S^*(t)}{\partial S_0} &= e^{jt} - e^{jt} \int_0^t e^{-jz} \cdot \frac{\partial c^*(z)}{\partial S_0} dz \\
 &= e^{jt} \left[1 - \int_0^t e^{-jz} \frac{g'(c^*(z))}{g''(c^*(z))\Psi(t^*)} dz \right] \\
 &= e^{jt} \left[1 - \frac{\Psi(t)}{\Psi(t^*)} \right].
 \end{aligned} \tag{3.49}$$

Interpretation of Sensitivity Analysis

Now that we have derived $\frac{\partial t^*}{\partial S_0}$, $\frac{\partial c^*(t)}{\partial S_0}$, and $\frac{\partial S^*(t)}{\partial S_0}$, we can see exactly how the time until depletion of wealth, the optimal consumption path, and the optimal accumulation

of wealth path are affected by a change in the initial wealth value:

$$\frac{\partial t^*}{\partial S_0} = \frac{1}{\Delta(t^*)\Psi(t^*)} \quad (3.50)$$

$$\frac{\partial c^*(t)}{\partial S_0} = \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \quad (3.51)$$

$$\frac{\partial S^*(t)}{\partial S_0} = e^{jt} \left[1 - \frac{\Psi(t)}{\Psi(t^*)} \right]. \quad (3.52)$$

Since the utility function, g , is defined to be a concave function where $g'(c) > 0$ and $g''(c) < 0$, we know that $\Psi(t^*) < 0$. From this, we can determine that $\frac{\partial t^*}{\partial S_0} > 0$ for $\Delta(t^*) < 0$. We can also easily see that $\frac{\partial c^*(t)}{\partial S_0} > 0$ for all $t \in [0, t^*)$. Since $\Psi(t^*) < \Psi(t) < 0$, we can make the observation that $0 < \frac{\Psi(t)}{\Psi(t^*)} < 1$. Therefore, $\frac{\partial S^*(t)}{\partial S_0} > 0$ for all $t \in [0, t^*)$. This tells us that if a consumer's initial wealth is increased, the optimal consumption path, the accumulated wealth, and the time until depletion of wealth will all increase as well.

3.2.2 Sensitivity of Model to Discount Factor α

Sensitivity of Depletion of Wealth Time t^*

The next step in the analysis is to study how changing the discount factor, α , affects the terminal wealth depletion time, optimal consumption path, and the associated optimal accumulation of wealth function. To do this, we will begin by taking the partial derivatives of S_0 , $c^*(t)$, and $S^*(t)$ with respect to α . The derivation of $\frac{\partial S_0}{\partial \alpha}$ to find $\frac{\partial t^*}{\partial \alpha}$ is

below:

$$\begin{aligned}
0 &= e^{-jt^*} \left[(g')^{-1} \left(\frac{e^{(j-\alpha)t^*} \Omega(t^*) g'(m(t^*))}{\Omega(t^*) e^{(j-\alpha)t^*}} \right) - m(t^*) \right] \frac{\partial t^*}{\partial \alpha} + \int_0^{t^*} e^{-jt} \frac{\partial (g')^{-1}(\sigma(t^*, t))}{\partial \alpha} dt \\
&= e^{-jt^*} [c^*(t^*) - m(t^*)] \frac{\partial t^*}{\partial \alpha} + \int_0^{t^*} e^{-jt} \frac{\partial (g')^{-1}(\sigma(t^*, t))}{\partial \alpha} dt \\
&= \int_0^{t^*} \frac{e^{-jt}}{g''((g')^{-1}(\sigma(t^*, t)))} \left(\frac{\Omega(t) e^{(j-\alpha)t} [e^{(j-\alpha)t^*} ((j-\alpha) \frac{\partial t^*}{\partial \alpha} - t^*) \Omega(t^*) g'(m(t^*))]}{(\Omega(t) e^{(j-\alpha)t})^2} + \right. \\
&\quad \left. + \frac{e^{(j-\alpha)t^*} (\Omega(t^*) (-\pi_{t^*}(t^*)) g'(m(t^*)) + \Omega(t^*) g''(m(t^*)) m'(t^*)) \frac{\partial t^*}{\partial \alpha}}{(\Omega(t) e^{(j-\alpha)t})^2} + \right. \\
&\quad \left. + \frac{t e^{(j-\alpha)t^*} \Omega(t^*) g'(m(t^*)) \Omega(t) e^{(j-\alpha)t}}{(\Omega(t) e^{(j-\alpha)t})^2} \right) dt \\
&= \int_0^{t^*} \frac{e^{-jt} \sigma(t^*, t)}{g''((g')^{-1}(\sigma(t^*, t)))} \left[\left(j - \alpha - \pi_{t^*}(t^*) + \frac{g''(m(t^*)) m'(t^*)}{g'(m(t^*))} \right) \frac{\partial t^*}{\partial \alpha} - (t^* - t) \right] dt \\
&= \int_0^{t^*} e^{-jt} \left[\frac{\Delta(t^*) g'(c^*(t))}{g''(c^*(t))} \cdot \frac{\partial t^*}{\partial \alpha} - \frac{(t - t^*) g'(c^*(t))}{g''(c^*(t))} \right] dt \\
&= \Delta(t^*) \Psi(t^*) \frac{\partial t^*}{\partial \alpha} - \int_0^{t^*} e^{-jt} (t^* - t) \frac{g'(c^*(t))}{g''(c^*(t))} dt \tag{3.53}
\end{aligned}$$

Solving equation (3.53) for $\frac{\partial t^*}{\partial \alpha}$ gives us

$$\frac{\partial t^*}{\partial \alpha} = \frac{\int_0^{t^*} e^{-jt} (t^* - t) \frac{g'(c^*(t))}{g''(c^*(t))} dt}{\Delta(t^*) \Psi(t^*)}. \tag{3.54}$$

Sensitivity of Optimal Consumption Path $c^*(t)$

The next step will be to derive $\frac{\partial c^*(t)}{\partial \alpha}$ for $t \in [0, t^*)$ from equation (3.37):

$$\begin{aligned}
\frac{\partial c^*(t)}{\partial \alpha} &= \frac{\Delta(t^*)g'(c^*(t))}{g''(c^*(t))} \cdot \frac{\partial t^*}{\partial \alpha} - \frac{(t^* - t)g'(c^*(t))}{g''(c^*(t))} \\
&= \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \int_0^{t^*} e^{-jz}(t^* - z) \frac{g'(c^*(z))}{g''(c^*(z))} dz - (t^* - t) \frac{g'(c^*(t))}{g''(c^*(t))} \\
&= \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \left[\int_0^{t^*} e^{-jz}(t^* - z) \frac{g'(c^*(z))}{g''(c^*(z))} dz - \Psi(t^*)(t^* - t) \right] \\
&= \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \left[\int_0^{t^*} e^{-jz}(t^* - z) \frac{g'(c^*(z))}{g''(c^*(z))} dz - \int_0^{t^*} e^{-jz} \frac{g'(c^*(z))}{g''(c^*(z))} dz (t^* - t) \right] \\
&= \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \int_0^{t^*} e^{-jz}(t - z) \frac{g'(c^*(z))}{g''(c^*(z))} dz \\
&= \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \left[t \int_0^{t^*} e^{-jz} \frac{g'(c^*(z))}{g''(c^*(z))} dz - \int_0^{t^*} e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz \right] \\
&= \frac{g'(c^*(t))}{g''(c^*(t))} \left[t\Psi(t^*) - \int_0^{t^*} e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz \right]. \tag{3.55}
\end{aligned}$$

Define $\xi = \frac{1}{\Psi(t^*)} \int_0^{t^*} e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz$. Then equation (3.55) becomes

$$\frac{\partial c^*(t)}{\partial \alpha} = \frac{g'(c^*(t))}{g''(c^*(t))} [t - \xi]. \tag{3.56}$$

Sensitivity of Optimal Wealth Path $S^*(t)$

The final derivation of $\frac{\partial S^*(t)}{\partial \alpha}$ for $t \in [0, t^*)$ is shown below:

$$\begin{aligned}
\frac{\partial S^*(t)}{\partial \alpha} &= -e^{jt} \int_0^t e^{-jx} \frac{\partial c^*(x)}{\partial \alpha} dx \\
&= -e^{jt} \int_0^t \left(e^{-jx} \frac{g'(c^*(x))}{g''(c^*(x))\Psi(t^*)} \cdot \int_0^{t^*} e^{-jz} (x-z) \frac{g'(c^*(z))}{g''(c^*(z))} dz \right) dx \\
&= \frac{e^{jt}}{\Psi(t^*)} \int_0^t \left(e^{-jx} \frac{g'(c^*(x))}{g''(c^*(x))} \cdot \int_0^{t^*} e^{-jz} (z-x) \frac{g'(c^*(z))}{g''(c^*(z))} dz \right) dx \\
&= \frac{e^{-jt}}{\Psi(t^*)} \int_0^t e^{-jx} \frac{g'(c^*(x))}{g''(c^*(x))} \left[\int_0^{t^*} e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz - x\Psi(t^*) \right] dx \\
&= \frac{e^{-jt}}{\Psi(t^*)} \left[\int_0^t e^{-jx} \frac{g'(c^*(x))}{g''(c^*(x))} dx \int_0^{t^*} e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz - \Psi(t^*) \int_0^t e^{-jx} x \frac{g'(c^*(x))}{g''(c^*(x))} dx \right] \\
&= e^{-jt} \left[\frac{\Psi(t)}{\Psi(t^*)} \int_0^{t^*} e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz - \int_0^t e^{-jx} x \frac{g'(c^*(x))}{g''(c^*(x))} dx \right]. \tag{3.57}
\end{aligned}$$

Interpretation of Sensitivity Analysis

We can now see from $\frac{\partial t^*}{\partial \alpha}$, $\frac{\partial c^*(t)}{\partial \alpha}$, and $\frac{\partial S^*(t)}{\partial \alpha}$ exactly how the time until depletion of wealth, the optimal consumption path, and the associated optimal accumulation of wealth function are affected by a change in the discount factor:

$$\frac{\partial t^*}{\partial \alpha} = \frac{\int_0^{t^*} e^{-jt} (t^* - t) \frac{g'(c^*(t))}{g''(c^*(t))} dt}{\Delta(t^*)\Psi(t^*)} \tag{3.58}$$

$$\frac{\partial c^*(t)}{\partial \alpha} = \frac{g'(c^*(t))}{g''(c^*(t))} [t - \xi] \tag{3.59}$$

$$\frac{\partial S^*(t)}{\partial \alpha} = e^{-jt} \left[\frac{\Psi(t)}{\Psi(t^*)} \int_0^{t^*} e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz - \int_0^t e^{-jx} x \frac{g'(c^*(x))}{g''(c^*(x))} dx \right]. \tag{3.60}$$

Just as before, the utility function, g , is defined to be a concave function where $g'(c) > 0$ and $g''(c) < 0$. This will again force $\Psi(t^*) < 0$. Notice then that $\frac{\partial t^*}{\partial \alpha}$ must be negative when $\Delta(t^*)$ is negative. Since $\frac{\partial t^*}{\partial \alpha} < 0$, we know that if we increase the discount factor α , the time until depletion of wealth will decrease. From equation (3.59), we can easily see that if $t \geq \xi$, then $\frac{\partial c^*(t)}{\partial \alpha} \leq 0$, and if $t \leq \xi$, then $\frac{\partial c^*(t)}{\partial \alpha} \geq 0$ for $t \in [0, t^*)$. This tells us

that an increase in the discount factor α will cause an individual's optimal consumption path to be greater before time ξ and lower after time ξ . Now notice that for $t \in (0, t^*)$, $\int_0^{t^*} e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz < \int_0^t e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz < 0$ and that $\frac{\Psi(t)}{\Psi(t^*)} > 0$. Then it is clear that $\frac{\partial S^*(t)}{\partial \alpha} < 0$ for $t \in (0, t^*)$. Therefore, if there is an increase in α , the optimal accumulation of wealth function will be lower [4].

3.3 Model Implementation

In this section, we will study a direct application of the model for a recently retired individual. Assume that the individual is currently 65 years old and has a constant income stream $m(t) = m$ from a retirement annuity. We will also assume a discount function of $\alpha(t) = e^{-\alpha t}$, an interest rate of $j(t) = j$, and a maximum possible lifetime of \bar{T} . Let us also define the utility function g with the Constant Relative Risk Aversion (CRRA) utility function:

$$g(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \gamma \neq 1 \\ \ln(c) & \gamma = 1 \end{cases}, \quad (3.61)$$

where c represents the consumption function and $\gamma \in \mathbb{R}$ represents the relative risk aversion coefficient [3]. By using the CRRA utility function, we can account for approximately how risk averse the individual is and how that may impact their optimal consumption and wealth paths. The higher the value of γ , the more risk averse the individual.

3.3.1 Sensitivity Analysis on the Relative Risk Aversion Coefficient

Before progressing deeper into the analysis, we must first rederive $c^*(t)$, $S^*(t)$, S_0 , $\Delta(t)$, and $\Psi(t)$ with our CRRA utility function g . Begin by noting that $g'(c) = c^{-\gamma}$, $(g')^{-1}(c) =$

$(1/c)^{1/\gamma}$, and $g''(c) = -\gamma c^{-\gamma-1}$ for all $\gamma \in \mathbb{R}$. The modified functions are shown below:

$$\begin{aligned}
c^*(t) &= (g)^{-1} \left(\frac{e^{(j-\alpha)t^*} \Omega(t^*) g'(m)}{\Omega(t) e^{(j-\alpha)t}} \right) \\
&= \left(\frac{\Omega(t) e^{(j-\alpha)t}}{e^{(j-\alpha)t^*} \Omega(t^*) (m)^{-\gamma}} \right)^{1/\gamma} \\
&= \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right)^{1/\gamma} \cdot m
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
S^*(t) &= e^{jt} \left[S_0 + \int_0^t e^{-jz} (m - c^*(z)) dz \right] \\
&= e^{jt} \left[S_0 + \int_0^t e^{-jz} \left(m - \left(\frac{\Omega(z)}{\Omega(t^*)} e^{(j-\alpha)(z-t^*)} \right) \right) dz \right]
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
S_0 &= \int_0^{t^*} e^{-jt} \left[(g')^{-1} \left(\frac{e^{(j-\alpha)t^*} \Omega(t^*) g'(m)}{\Omega(t) e^{(j-\alpha)t}} \right) - m \right] dt \\
&= \int_0^{t^*} e^{-jt} \left[\left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right)^{1/\gamma} \cdot m - m \right] dt
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
\Delta(t) &= -\pi_t(t) - \alpha + j - \frac{m'(t) g''(m)}{g'(m)} \\
&= -\pi_t(t) - \alpha + j
\end{aligned} \tag{3.65}$$

$$\begin{aligned}
\Psi(t) &= \int_0^t e^{-jz} \left[\frac{g'(c^*(z))}{g''(c^*(z))} \right] dz \\
&= \int_0^t e^{-jz} \left[\frac{c^*(z)^{-\gamma}}{-\gamma c^*(z)^{-\gamma-1}} \right] dz \\
&= \frac{-1}{\gamma} \int_0^t e^{-jz} c^*(z) dz.
\end{aligned} \tag{3.66}$$

We may now begin our analysis by deriving the partial derivatives of t^* , $c^*(t)$, and $S^*(t)$ with respect to the relative risk aversion coefficient γ . This will help us determine

how sensitive the terminal depletion of wealth time, optimal consumption path, and optimal accumulation of wealth path are all affected by a small change in γ .

Sensitivity of Depletion of Wealth Time t^*

The derivation of $\frac{\partial t^*}{\partial \gamma}$ will involve taking the partial derivative of equation (3.64) with respect to γ :

$$\begin{aligned}
0 &= e^{-j t^*} \left[\left(\frac{\Omega(t^*)}{\Omega(t^*)} e^{(j-\alpha)(t^*-t^*)} \right)^{1/\gamma} m - m \right] \left(\frac{\partial t^*}{\partial \gamma} \right) + \\
&\quad + \int_0^{t^*} e^{-j t} \left[\frac{\partial}{\partial \gamma} \left(\left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right)^{1/\gamma} \right) m - m \right] dt \\
&= \int_0^{t^*} e^{-j t} \left[\frac{\partial}{\partial \gamma} \left(\left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right)^{1/\gamma} \right) m - m \right] dt. \tag{3.67}
\end{aligned}$$

Before moving any further, we must use logarithmic differentiation to find the partial derivative of $\left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right)^{1/\gamma}$ with respect to γ :

$$\begin{aligned}
y &= \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right)^{1/\gamma} \\
\ln y &= \frac{1}{\gamma} \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) \\
\frac{\partial}{\partial \gamma} (\ln y) &= \frac{\partial}{\partial \gamma} \left(\frac{1}{\gamma} \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) \right) \\
\frac{1}{y} \cdot \frac{\partial y}{\partial \gamma} &= \frac{-1}{\gamma^2} \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) + \\
&\quad + \frac{1}{\gamma} \cdot \frac{1}{\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)}} \left[\frac{\Omega(t)(\pi_{t^*}(t^*))\Omega(t^*)}{(\Omega(t^*))^2} e^{(j-\alpha)(t-t^*)} + \right. \\
&\quad \left. + (\alpha - j) \frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right] \left(\frac{\partial t^*}{\partial \gamma} \right) \\
\frac{1}{y} \cdot \frac{\partial y}{\partial \gamma} &= \frac{-1}{\gamma^2} \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) + \frac{1}{\gamma} (\pi_{t^*}(t^*) + \alpha - j) \left(\frac{\partial t^*}{\partial \gamma} \right) \tag{3.68}
\end{aligned}$$

Solving equation (3.68) for $\frac{\partial y}{\partial \gamma}$ gives us the derivative of $\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}$ with respect to γ :

$$\frac{\partial y}{\partial \gamma} = \left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma} \left[\frac{-1}{\gamma^2} \ln \left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)} \right) - \frac{\Delta(t^*)}{\gamma} \left(\frac{\partial t^*}{\partial \gamma} \right) \right]. \quad (3.69)$$

Now that we have derived the partial derivative of $\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}$ with respect to γ (equation (3.69)), we may now complete the derivation of $\frac{\partial t^*}{\partial \gamma}$. Continuing from equation (3.67), we have

$$\begin{aligned} 0 &= \int_0^{t^*} e^{-jt} \left[\frac{\partial}{\partial \gamma} \left(\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)} \right)^{1/\gamma} m - m \right) \right] dt \\ &= \int_0^{t^*} e^{-jt} m \left[\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)} \right)^{1/\gamma} \left[\frac{-1}{\gamma^2} \ln \left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)} \right) - \frac{\Delta(t^*)}{\gamma} \left(\frac{\partial t^*}{\partial \gamma} \right) \right] \right] dt \\ &= \frac{-1}{\gamma^2} \int_0^{t^*} e^{-jt} \cdot c^*(t) \ln \left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)} \right) dt - \frac{\Delta(t^*)}{\gamma} \left(\frac{\partial t^*}{\partial \gamma} \right) \int_0^{t^*} e^{-jt} c^*(t) dt \\ &= \frac{-1}{\gamma^2} \int_0^{t^*} e^{-jt} \cdot c^*(t) \ln \left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)} \right) dt + \Delta(t^*) \Psi(t^*) \left(\frac{\partial t^*}{\partial \gamma} \right). \end{aligned} \quad (3.70)$$

Thus,

$$\frac{\partial t^*}{\partial \gamma} = \frac{\int_0^{t^*} e^{-jt} c^*(t) \ln \left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)} \right) dt}{\gamma^2 \Delta(t^*) \Psi(t^*)}. \quad (3.71)$$

Sensitivity of Optimal Consumption Path $\mathbf{c}^*(\mathbf{t})$

By using equation (3.69), we may also derive $\frac{\partial \mathbf{c}^*(\mathbf{t})}{\partial \gamma}$:

$$\begin{aligned}
\frac{\partial \mathbf{c}^*(\mathbf{t})}{\partial \gamma} &= m \left(\frac{\Omega(\mathbf{t})}{\Omega(\mathbf{t}^*)} e^{(j-\alpha)(\mathbf{t}-\mathbf{t}^*)} \right)^{1/\gamma} \left[\frac{-1}{\gamma^2} \ln \left(\frac{\Omega(\mathbf{t})}{\Omega(\mathbf{t}^*)} e^{(j-\alpha)(\mathbf{t}-\mathbf{t}^*)} \right) - \frac{\Delta(\mathbf{t}^*)}{\gamma} \left(\frac{\partial \mathbf{t}^*}{\partial \gamma} \right) \right] \\
&= \frac{-\mathbf{c}^*(\mathbf{t})}{\gamma^2} \ln \left(\frac{\Omega(\mathbf{t})}{\Omega(\mathbf{t}^*)} e^{(j-\alpha)(\mathbf{t}-\mathbf{t}^*)} \right) - \Delta(\mathbf{t}^*) \left(\frac{\mathbf{c}^*(\mathbf{t})}{\gamma} \right) \left(\frac{\partial \mathbf{t}^*}{\partial \gamma} \right) \\
&= \frac{-\mathbf{c}^*(\mathbf{t})}{\gamma^2} \ln \left(\frac{\Omega(\mathbf{t})}{\Omega(\mathbf{t}^*)} e^{(j-\alpha)(\mathbf{t}-\mathbf{t}^*)} \right) + \\
&\quad - \Delta(\mathbf{t}^*) \left(\frac{\mathbf{c}^*(\mathbf{t})}{\gamma} \right) \left(\frac{\int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) \ln \left(\frac{\Omega(z)}{\Omega(\mathbf{t}^*)} e^{(j-\alpha)(z-\mathbf{t}^*)} \right) dz}{\gamma^2 \Delta(\mathbf{t}^*) \Psi(\mathbf{t}^*)} \right) \\
&= \frac{-\mathbf{c}^*(\mathbf{t})}{\gamma^2} \ln \left(\frac{\Omega(\mathbf{t})}{\Omega(\mathbf{t}^*)} e^{(j-\alpha)(\mathbf{t}-\mathbf{t}^*)} \right) + \frac{\mathbf{c}^*(\mathbf{t}) \int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) \ln \left(\frac{\Omega(z)}{\Omega(\mathbf{t}^*)} e^{(j-\alpha)(z-\mathbf{t}^*)} \right) dz}{\gamma^2 \int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) dz} \\
&= \mathbf{c}^*(\mathbf{t}) \left[\frac{-\int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) dz \cdot \ln \left(\frac{\Omega(\mathbf{t})}{\Omega(\mathbf{t}^*)} e^{(j-\alpha)(\mathbf{t}-\mathbf{t}^*)} \right)}{\gamma \int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) dz} + \right. \\
&\quad \left. + \frac{\int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) \ln \left(\frac{\Omega(z)}{\Omega(\mathbf{t}^*)} e^{(j-\alpha)(z-\mathbf{t}^*)} \right) dz}{\gamma^2 \int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) dz} \right] \\
&= \mathbf{c}^*(\mathbf{t}) \left[\frac{-\int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) dz \cdot (\ln \Omega(\mathbf{t}) - \ln \Omega(\mathbf{t}^*) + j\mathbf{t} - j\mathbf{t}^* - \alpha\mathbf{t} + \alpha\mathbf{t}^*)}{\gamma^2 \int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) dz} + \right. \\
&\quad \left. + \frac{\int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) (\ln \Omega(z) - \ln \Omega(\mathbf{t}^*) + jz - j\mathbf{t}^* - \alpha z + \alpha\mathbf{t}^*) dz}{\gamma^2 \int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) dz} \right]. \tag{3.72}
\end{aligned}$$

Through just a bit of simplification, we can rewrite equation (3.72) as the following:

$$\frac{\partial \mathbf{c}^*(\mathbf{t})}{\partial \gamma} = \mathbf{c}^*(\mathbf{t}) \left[\frac{\int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) \ln \left(\frac{\Omega(z)}{\Omega(\mathbf{t})} e^{(j-\alpha)(z-\mathbf{t})} \right) dz}{\gamma^2 \int_0^{\mathbf{t}^*} e^{-jz} \mathbf{c}^*(z) dz} \right]. \tag{3.73}$$

Sensitivity of Optimal Wealth Path $S^*(t)$

The derivation of $\frac{\partial S^*(t)}{\partial \gamma}$ is fairly straightforward and is shown below:

$$\begin{aligned}
 \frac{\partial S^*(t)}{\partial \gamma} &= -e^{jt} \int_0^t e^{-jx} \frac{\partial c^*(x)}{\partial \gamma} dx \\
 &= -e^{jt} \int_0^t e^{-jx} \left(\frac{c^*(x) \int_0^{t^*} e^{-jz} c^*(z) \ln \left(\frac{\Omega(z)}{\Omega(x)} e^{(j-\alpha)(z-x)} \right) dz}{\gamma^2 \int_0^{t^*} e^{-jz} c^*(z) dz} \right) dx \\
 &= \frac{-e^{jt} \int_0^t e^{-jx} c^*(x) \int_0^{t^*} e^{-jz} c^*(z) \ln \left(\frac{\Omega(z)}{\Omega(x)} e^{(j-\alpha)(z-x)} \right) dz dx}{\gamma^2 \int_0^{t^*} e^{-jz} c^*(z) dz}.
 \end{aligned} \tag{3.74}$$

Interpretation of Sensitivity Analysis

From our derivations of $\frac{\partial t^*}{\partial \gamma}$, $\frac{\partial c^*(t)}{\partial \gamma}$, and $\frac{\partial S^*(t)}{\partial \gamma}$, we can see how the time until depletion of wealth, optimal consumption path, and optimal accumulation path are affected by a change in the relative risk aversion coefficient γ .

$$\frac{\partial t^*}{\partial \gamma} = \frac{\int_0^{t^*} e^{-jt} c^*(t) \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) dt}{\gamma^2 \Delta(t^*) \Psi(t^*)} \tag{3.75}$$

$$\frac{\partial c^*(t)}{\partial \gamma} = c^*(t) \left[\frac{\int_0^{t^*} e^{-jz} c^*(z) \ln \left(\frac{\Omega(z)}{\Omega(t)} e^{(j-\alpha)(z-t)} \right) dz}{\gamma^2 \int_0^{t^*} e^{-jz} c^*(z) dz} \right] \tag{3.76}$$

$$\frac{\partial S^*(t)}{\partial \gamma} = \frac{-e^{jt} \int_0^t e^{-jx} c^*(x) \int_0^{t^*} e^{-jz} c^*(z) \ln \left(\frac{\Omega(z)}{\Omega(x)} e^{(j-\alpha)(z-x)} \right) dz dx}{\gamma^2 \int_0^{t^*} e^{-jz} c^*(z) dz}. \tag{3.77}$$

Notice that $\Omega(t)e^{(j-\alpha)t} > \Omega(t^*)e^{(j-\alpha)t^*}$ for $t \in [0, t^*)$ and that $\pi'_t(t) > 0$ for $t \in [0, t^*)$.

Given the assumptions of the model and recalling that $\Psi(t^*) \leq 0$, we can see from equation (3.75) that $\frac{\partial t^*}{\partial \gamma}$ must be positive when $\Delta(t^*)$ is negative. This tells us that if

we are more risk averse our time until depletion of wealth will increase. Now let us

define ω as the time at which $\int_0^{t^*} e^{-jz} c^*(z) \ln \left(\frac{\Omega(z)}{\Omega(\omega)} e^{(j-\alpha)(z-\omega)} \right) dz = 0$ for $c^* > 0$. Then

$\frac{\partial c^*(t)}{\partial \gamma} \geq 0$ if $t \geq \omega$ and $\frac{\partial c^*(t)}{\partial \gamma} \leq 0$ if $t \leq \omega$. This indicates that individuals who have a

higher relative risk aversion coefficient tend to consume less before time ω and consume

more afterwards. The opposite is true for those who are less risk averse. We can also see that $\frac{\partial S^*(t)}{\partial \gamma} > 0$ for $t \in (0, t^*)$, meaning individuals that are more risk averse will retain their wealth longer than individuals that are less risk averse.

3.3.2 MATLAB Simulation

The Gompertz probability distribution is a commonly used distribution in actuarial mortality models to determine the probability of survival and death. For this reason, we will integrate the Gompertz probability density function,

$$f(x) = ae^{bx}e^{-\frac{a}{b}(e^{bx}-1)}, \quad (3.78)$$

so that we can properly define our survival function $\Omega(t)$ [7]. Since $\Omega(t)$ is defined to be the probability of an individual surviving past a certain time, we will integrate the Gompertz probability density function in the following manner:

$$\begin{aligned} \Omega(t) &= 1 - \int_0^t ae^{bx}e^{-\frac{a}{b}(e^{bx}-1)} dx \\ &= 1 - \left(-e^{-\frac{a}{b}(e^{bx}-1)} \Big|_0^t \right) \\ &= e^{-\frac{a}{b}(e^{bt}-1)}. \end{aligned} \quad (3.79)$$

For the purpose of simulating our model in MATLAB, we will set $a = 0.000081$ and $b = 0.087$, just as was done in Leung's 1994 paper, "Uncertain Lifetime, the Theory of the Consumer, and the Life Cycle Hypothesis." In the simulated model, we will also be using a selection of the parameters from the same paper shown in Table 3.1 [3].

		t^*		
		$\underline{S(65)/m}$		
$\underline{\gamma}$	$\underline{\alpha}$	$\underline{1}$	$\underline{5}$	$\underline{10}$
3	0.10	73	80	84
	0.05	75	83	88
	0.03	77	85	89
	0.01	79	87	91
1	0.10	70	74	76
	0.05	71	77	80
	0.03	73	79	82
	0.01	75	81	84
0.5	0.10	68	71	73
	0.05	70	73	76
	0.03	71	75	78
	0.01	73	78	80

Table 3.1: A selection of parameters from Leung's "Uncertain Lifetime, The Theory of the Consumer, and the Life Cycle Hypothesis"

For model simplicity, we will define $m = 1$. Displayed below are figures showing the simulated optimal consumption and associated accumulated wealth functions for each of the parameters in Table 3.1 and for $j = 0.03$.

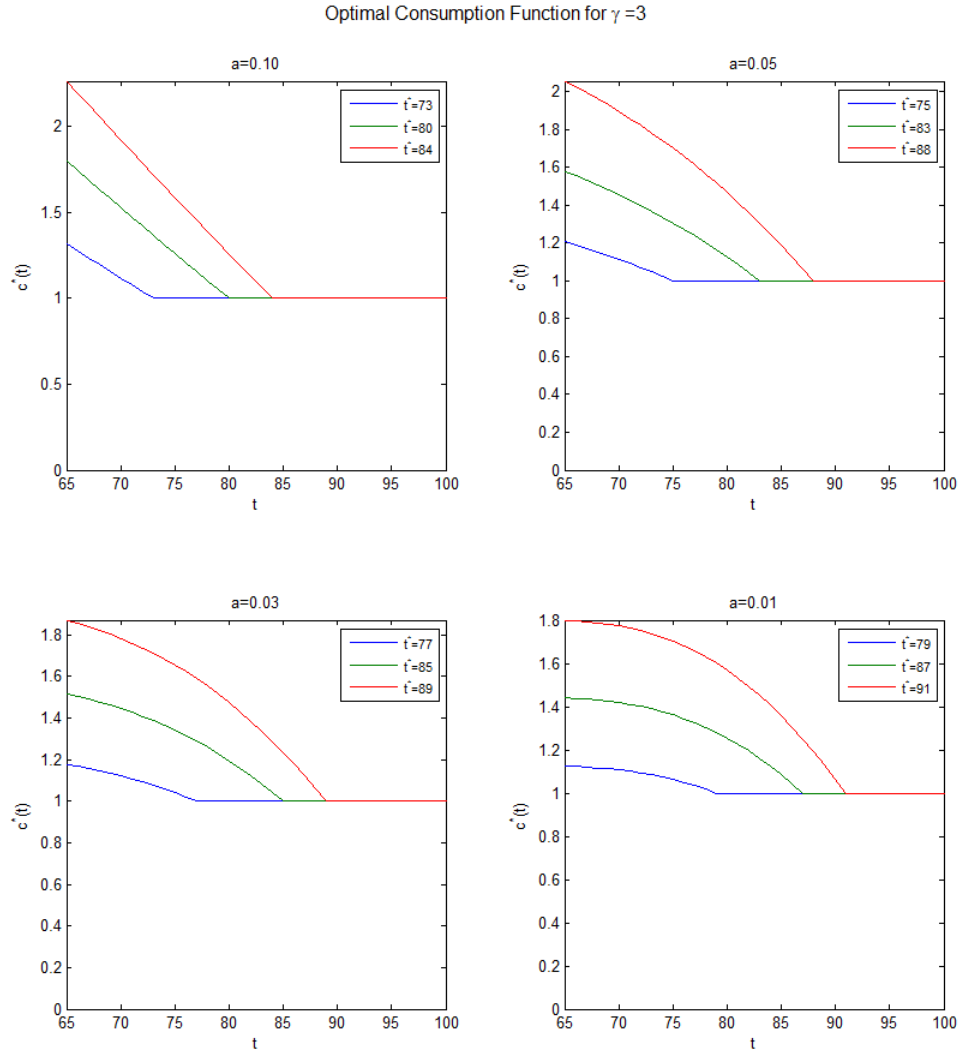


Figure 3.1: Each plot below displays the optimal consumption path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 3$.

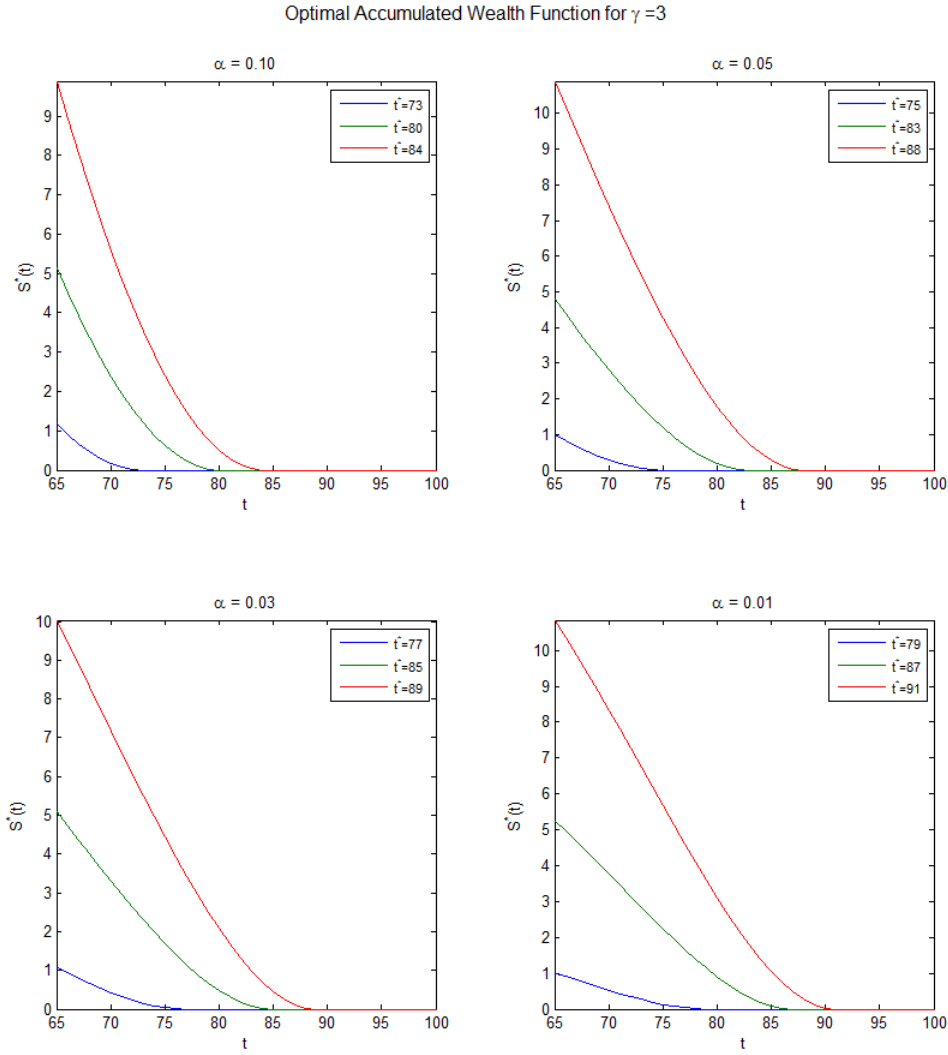


Figure 3.2: Each plot below displays the associated optimal accumulation of wealth path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 3$.

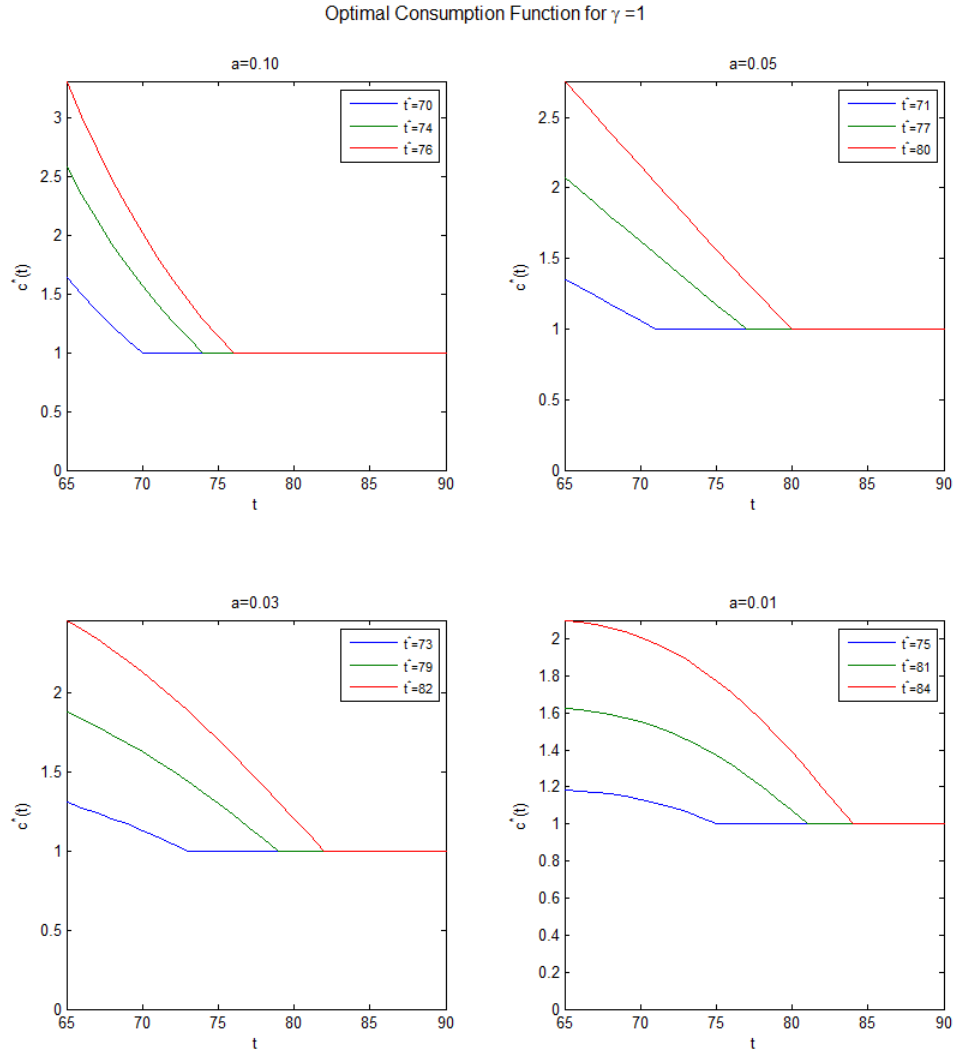


Figure 3.3: Each plot below displays the optimal consumption path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 1$.

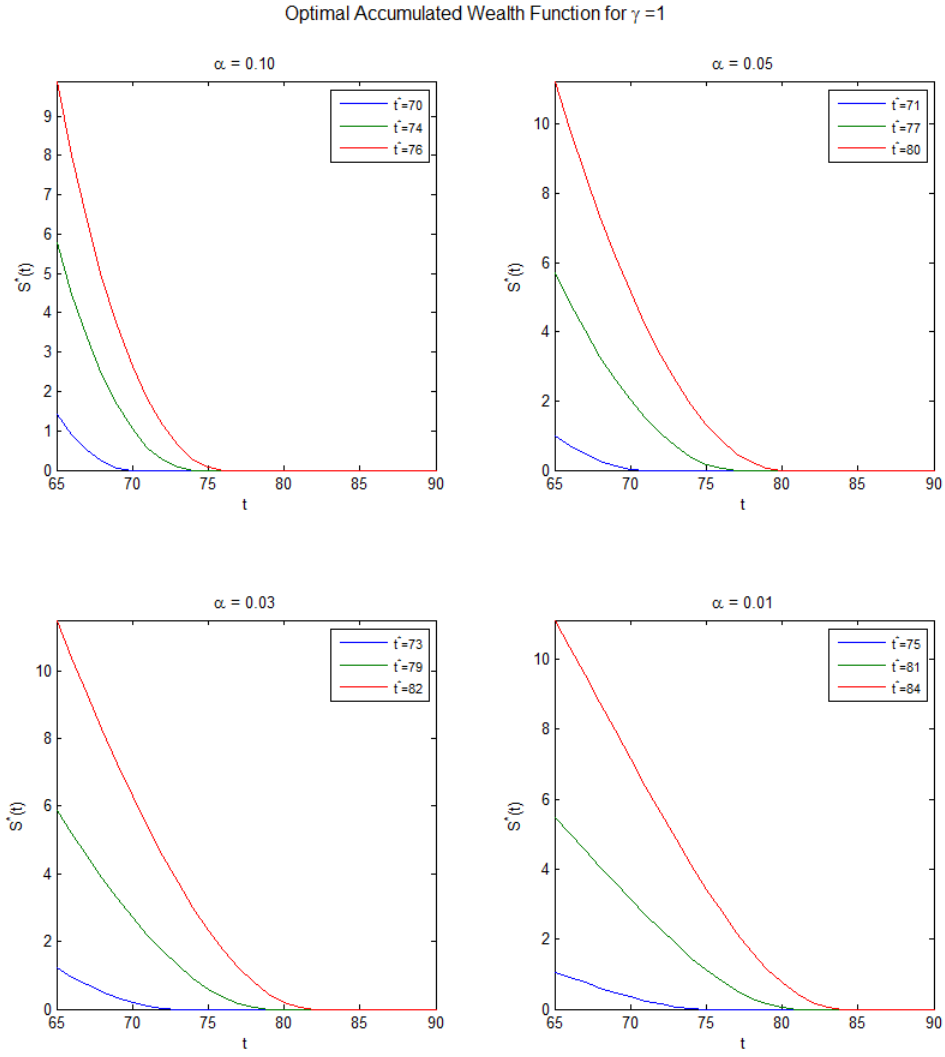


Figure 3.4: Each plot below displays the associated optimal accumulation of wealth path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 1$.

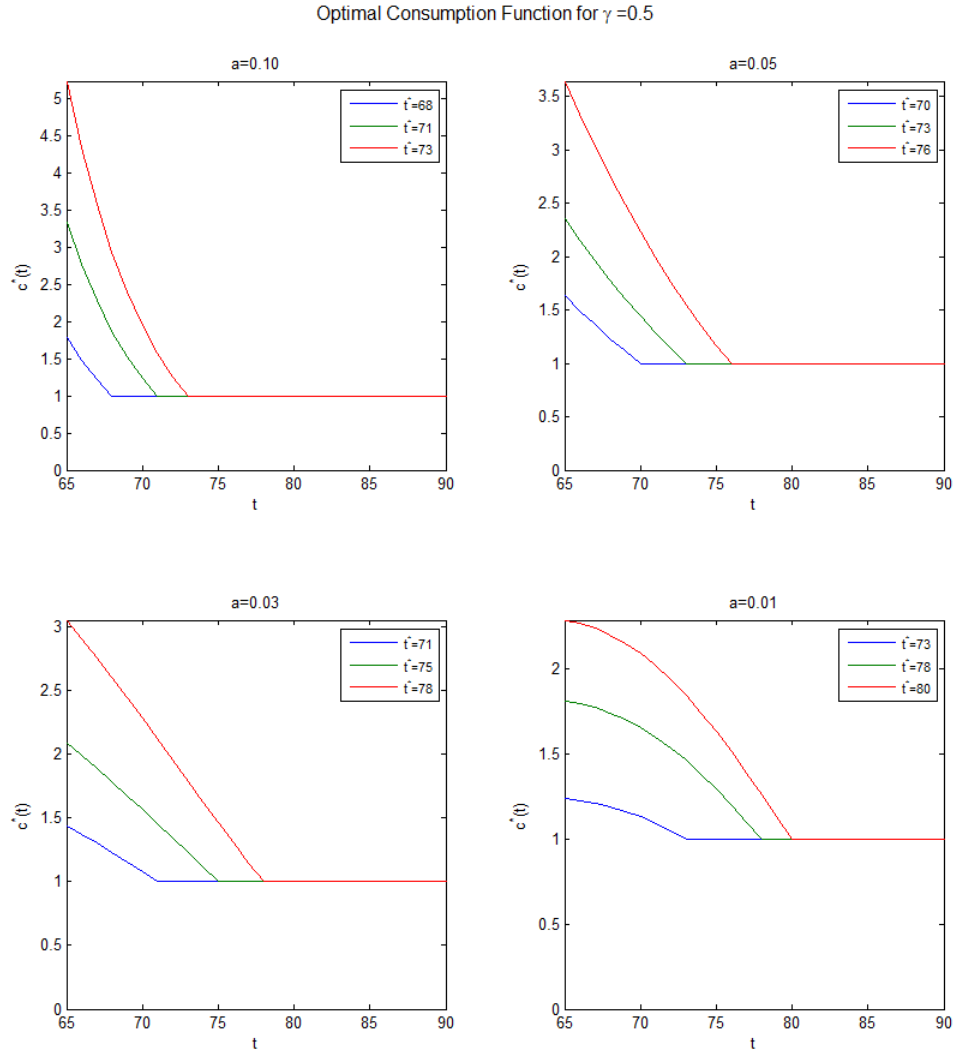


Figure 3.5: Each plot below displays the optimal consumption path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 0.5$.

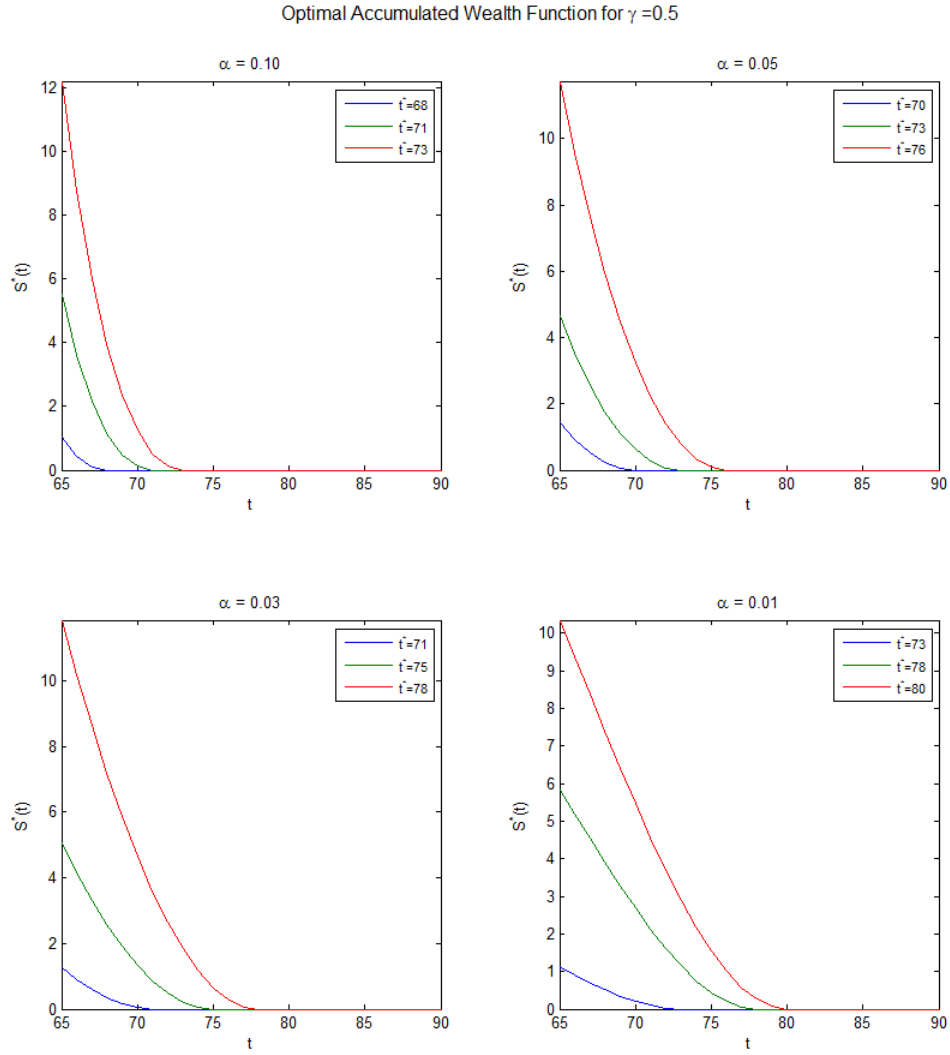


Figure 3.6: Each plot below displays the associated optimal accumulation of wealth path for an individual with a specified t^* found from the equation for S_0 , $\alpha = 0.10, 0.05, 0.03$, and 0.01 , and $\gamma = 0.5$.

Observe from figures 3.1, 3.3, and 3.5 that when γ is lower in value, the steeper the consumption function. This indicates that when a consumer is less risk averse and perhaps even risk seeking, the consumer will tend to spend more money right after retirement. On the other hand, if a consumer is very risk averse, they will tend to spend quite a bit less money right after retirement. The dynamics of the accumulation of wealth

functions go hand-in-hand with the dynamics of the consumption functions. We can see from figures 3.2, 3.4, and 3.6 that when γ is lower in value, the accumulation of wealth function decreases at a much quicker rate than when γ is greater in value. With this, we can also see that the depletion of wealth time t^* is much lower for a small value of γ than with a large value of γ . In other words, an individual tends to ration their savings when they are more risk averse, causing their time until depletion of wealth to increase.

Chapter 4

Conclusion

Throughout the course of this thesis, we discussed the mathematics behind optimal control theory and Pontryagin's Maximum Principle, as well as applied optimal control theory to the life-cycle savings model under uncertain lifetime. From our sensitivity analysis, we are able to conclude that an individual with a higher initial wealth value will have a higher optimal consumption path and accumulated wealth path. We also found that the changes in dynamics of the consumption path when increasing the discount factor were not monotonic. In fact if the discount factor is increased, then the individual will have a higher optimal consumption path until a certain point in time. Once that time is reached, the individual's optimal consumption path will be lower. An increase in the discount factor also resulted in a decrease in the accumulation of wealth function. By simulating the model in MATLAB, we are able to not only visualize the dynamics of the model, but we are also able to verify some of the findings from the sensitivity analysis. From this paper, we can see exactly how we can control our consumption when given an initial wealth, income function, interest rate, and discount factor so that we can optimize the Fisher Utility function over an entire lifetime.

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Appendix 1

1 MATLAB Code For Optimal Consumption Function

```
1 j=0.03;
2 gamma=[3 1 0.5 0.1];
3 a=[0.10 0.05 0.03 0.01];
4 tstar=[73 80 84; 75 83 88; 77 85 89; 79 87 91; 70 74 76; 71 77
        80; 73 79 82; 75 81 84; 68 71 73; 70 73 76; 71 75 78; 73 78
        80; 66 67 68; 67 69 70; 68 70 71; 69 72 73];
5 m=1;
6
7 t=linspace(65,140,100);
8
9 figure
10 for i=1:1 %gamma
11     for q=1:1 %a
12         for k=1:3 %columns of tstar
13             t=ones(76,1);
14             y=ones(76,1);
15             for p=1:76
16                 t(p)=p+64;
```

```

17         if (t(p) < tstar(4.*(i-1)+q,k))
18             y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                    exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                    ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                    +q,k))))^(1/gamma(i))*m
19         else
20             y(p)=m
21         end
22     end
23     subplot(2,2,1)
24     plot(t,y)
25     hold all
26     legend({'t^*=73','t^*=80','t^*=84'},'FontSize',8)
27     title('a=0.10')
28     xlabel('t')
29     ylabel('c^*(t)')
30     axis([65 100 0 inf])
31 end
32 end
33 end
34
35 for i=1:1 %gamma
36     for q=2:2 %a
37         for k=1:3 %columns of tstar
38             t=ones(76,1);
39             y=ones(76,1);
40             for p=1:76

```

```

41         t(p)=p+64;
42         if (t(p) < tstar(4.*(i-1)+q,k))
43             y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                    exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                    )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                    +q,k))))^(1/gamma(i))*m
44         else
45             y(p)=m
46         end
47     end
48     subplot(2,2,2)
49     plot(t,y)
50     hold all
51     legend({'t^*=75','t^*=83','t^*=88'},'FontSize',8)
52     xlabel('t')
53     ylabel('c^*(t)')
54     title('a=0.05')
55     axis([65 100 0 inf])
56 end
57 end
58 end
59
60 for i=1:1 %gamma
61     for q=3:3 %a
62         for k=1:3 %columns of tstar
63             t=ones(76,1);
64             y=ones(76,1);

```



```

65         for p=1:76
66             t(p)=p+64;
67             if (t(p) < tstar(4.*(i-1)+q,k))
68                 y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                    exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                    )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                    +q,k))))^(1/gamma(i))*m
69             else
70                 y(p)=m
71             end
72         end
73         subplot(2,2,3)
74         plot(t,y)
75         hold all
76         legend({'t^*=77','t^*=85','t^*=89'},'FontSize',8)
77         xlabel('t')
78         ylabel('c^*(t)')
79         title('a=0.03')
80         axis([65 100 0 inf])
81     end
82 end
83 end
84
85 for i=1:1 %gamma
86     for q=4:4 %a
87         for k=1:3 %columns of tstar
88             t=ones(76,1);

```

```

89         y=ones(76,1);
90         for p=1:76
91             t(p)=p+64;
92             if (t(p) < tstar(4.*(i-1)+q,k))
93                 y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                     exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                     )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                     +q,k))))^(1/gamma(i))*m
94             else
95                 y(p)=m
96             end
97         end
98         subplot(2,2,4)
99         plot(t,y)
100        hold all
101        legend({'t^*=79','t^*=87','t^*=91'},'FontSize',8)
102        xlabel('t')
103        ylabel('c^*(t)')
104        title('a=0.01')
105        axis([65 100 0 inf])
106    end
107end
108end
109 annotation('textbox',[0 0.9 1 0.1], 'String', 'Optimal
    Consumption Function for \gamma =3', 'EdgeColor', 'none', '
    HorizontalAlignment', 'center', 'FontSize',12)
110 figure

```

```

111 for i=2:2 %gamma
112     for q=1:1 %a
113         for k=1:3 %columns of tstar
114             t=ones(76,1);
115             y=ones(76,1);
116             for p=1:76
117                 t(p)=p+64;
118                 if (t(p) < tstar(4.*(i-1)+q,k))
119                     y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
120                         exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
121                             )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
122                             +q,k))))^(1/gamma(i))*m
123                     else
124                         y(p)=m
125                     end
126             end
127         end
128     subplot(2,2,1)
129     plot(t,y)
130     hold all
131     legend({'t^*=70','t^*=74','t^*=76'},'FontSize',8)
132     title('a=0.10')
133     xlabel('t')
134     ylabel('c^*(t)')
135     axis([65 90 0 inf])
136 end
137 end
138 end

```

```

135
136 for i=2:2 %gamma
137     for q=2:2 %a
138         for k=1:3 %columns of tstar
139             t=ones(76,1);
140             y=ones(76,1);
141             for p=1:76
142                 t(p)=p+64;
143                 if (t(p) < tstar(4.*(i-1)+q,k))
144                     y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
145                         exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k)
146                             )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
147                             +q,k))))^(1/gamma(i))*m
148                 else
149                     y(p)=m
150                 end
151             end
152         end
153     end
154     subplot(2,2,2)
155     plot(t,y)
156     hold all
157     legend({'t^*=71','t^*=77','t^*=80'},'FontSize',8)
158     title('a=0.05')
159     xlabel('t')
160     ylabel('c^*(t)')
161     axis([65 90 0 inf])
162 end
end

```

```

159 end
160
161 for i=2:2 %gamma
162     for q=3:3 %a
163         for k=1:3 %columns of tstar
164             t=ones(76,1);
165             y=ones(76,1);
166             for p=1:76
167                 t(p)=p+64;
168                 if (t(p) < tstar(4.*(i-1)+q,k))
169                     y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
170                         exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k)
171                             )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
172                             +q,k))))^(1/gamma(i))*m
173                 else
174                     y(p)=m
175                 end
176             end
177         end
178     end
179     subplot(2,2,3)
180     plot(t,y)
181     hold all
182     legend({'t^*=73','t^*=79','t^*=82'},'FontSize',8)
183     title('a=0.03')
184     xlabel('t')
185     ylabel('c^*(t)')
186     axis([65 90 0 inf])
187 end

```

```

183     end
184 end
185
186 for i=2:2 %gamma
187     for q=4:4 %a
188         for k=1:3 %columns of tstar
189             t=ones(76,1);
190             y=ones(76,1);
191             for p=1:76
192                 t(p)=p+64;
193                 if (t(p) < tstar(4.*(i-1)+q,k))
194                     y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
195                         exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
196                             ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
197                             +q,k))))^(1/gamma(i))*m
198                 else
199                     y(p)=m
200             end
201         end
202     end
203     subplot(2,2,4)
204     plot(t,y)
205     hold all
206     legend({'t^*=75','t^*=81','t^*=84'},'FontSize',8)
207     title('a=0.01')
208     xlabel('t')
209     ylabel('c^*(t)')
210     axis([65 90 0 inf])

```

```

207         end
208     end
209 end
210 annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal
    Consumption Function for \gamma =1', 'EdgeColor', 'none', '
    HorizontalAlignment', 'center', 'FontSize', 12)
211
212
213
214
215 figure
216 for i=3:3 %gamma
217     for q=1:1 %a
218         for k=1:3 %columns of tstar
219             t=ones(76,1);
220             y=ones(76,1);
221             for p=1:76
222                 t(p)=p+64;
223                 if (t(p) < tstar(4.*(i-1)+q,k))
224                     y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                        exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k)
                        )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                        +q,k))))^(1/gamma(i))*m
225                 else
226                     y(p)=m
227                 end
228             end
229         end
230     end
231 end

```

```

229         subplot(2,2,1)
230         plot(t,y)
231         hold all
232         legend({'t^*=68','t^*=71','t^*=73'},'FontSize',8)
233         title('a=0.10')
234         xlabel('t')
235         ylabel('c^*(t)')
236         axis([65 90 0 inf])
237     end
238 end
239 end
240
241 for i=3:3 %gamma
242     for q=2:2 %a
243         for k=1:3 %columns of tstar
244             t=ones(76,1);
245             y=ones(76,1);
246             for p=1:76
247                 t(p)=p+64;
248                 if (t(p) < tstar(4.*(i-1)+q,k))
249                     y(p)=((((exp(-0.00093*(exp(0.087*t(p))-1)))/(
250                         exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k)
251                             )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
252                             +q,k))))^(1/gamma(i))*m
253
254                 else
255                     y(p)=m
256                 end
257             end
258         end
259     end
260 end

```



```

253         end
254         subplot(2,2,2)
255         plot(t,y)
256         hold all
257         legend({'t^*=70','t^*=73','t^*=76'},'FontSize',8)
258         title('a=0.05')
259         xlabel('t')
260         ylabel('c^*(t)')
261         axis([65 90 0 inf])
262     end
263 end
264 end
265
266 for i=3:3 %gamma
267     for q=3:3 %a
268         for k=1:3 %columns of tstar
269             t=ones(76,1);
270             y=ones(76,1);
271             for p=1:76
272                 t(p)=p+64;
273                 if (t(p) < tstar(4.*(i-1)+q,k))
274                     y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
275                         exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k)
276                         ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
277                         +q,k))))^(1/gamma(i))*m
278                 else
279                     y(p)=m

```

```

277         end
278     end
279     subplot(2,2,3)
280     plot(t,y)
281     hold all
282     legend({'t^*=71','t^*=75','t^*=78'},'FontSize',8)
283     title('a=0.03')
284     xlabel('t')
285     ylabel('c^*(t)')
286     axis([65 90 0 inf])
287 end
288 end
289 end
290
291 for i=3:3 %gamma
292     for q=4:4 %a
293         for k=1:3 %columns of tstar
294             t=ones(76,1);
295             y=ones(76,1);
296             for p=1:76
297                 t(p)=p+64;
298                 if (t(p) < tstar(4.*(i-1)+q,k))
299                     y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
300                         exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k)
301                             )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
302                             +q,k))))^(1/gamma(i))*m
303                 else

```

```

301         y(p)=m
302     end
303 end
304 subplot(2,2,4)
305 plot(t,y)
306 hold all
307 legend({'t^*=73','t^*=78','t^*=80'},'FontSize',8)
308 title('a=0.01')
309 xlabel('t')
310 ylabel('c^*(t)')
311 axis([65 90 0 inf])
312 end
313 end
314 end
315 annotation('textbox',[0 0.9 1 0.1],'String','Optimal
Consumption Function for \gamma =0.5','EdgeColor','none','
HorizontalAlignment','center','FontSize',12)
316
317
318
319
320
321 figure
322 for i=4:4 %gamma
323     for q=1:1 %a
324         for k=1:3 %columns of tstar
325             t=ones(76,1);

```

```

326         y=ones(76,1);
327         for p=1:76
328             t(p)=p+64;
329             if (t(p) < tstar(4.*(i-1)+q,k))
330                 y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
331                     exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
332                         )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
333                         +q,k))))^(1/gamma(i))*m
334             else
335                 y(p)=m
336             end
337         end
338         subplot(2,2,1)
339         plot(t,y)
340         hold all
341         legend({'t^*=66','t^*=67','t^*=68'},'FontSize',8)
342         title('a=0.10')
343         xlabel('t')
344         ylabel('c^*(t)')
345         axis([65 80 0 inf])
346     end
347 end
348 end
349 for i=4:4 %gamma
350     for q=2:2 %a
351         for k=1:3 %columns of tstar

```

```

350         t=ones(76,1);
351         y=ones(76,1);
352         for p=1:76
353             t(p)=p+64;
354             if (t(p) < tstar(4.*(i-1)+q,k))
355                 y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                    exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                    )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                    +q,k))))^(1/gamma(i))*m
356             else
357                 y(p)=m
358             end
359         end
360         subplot(2,2,2)
361         plot(t,y)
362         hold all
363         legend({'t^*=67','t^*=69','t^*=70'},'FontSize',8)
364         title('a=0.05')
365         xlabel('t')
366         ylabel('c^*(t)')
367         axis([65 80 0 inf])
368     end
369 end
370 end
371
372 for i=4:4 %gamma
373     for q=3:3 %a

```

```

374     for k=1:3 %columns of tstar
375         t=ones(76,1);
376         y=ones(76,1);
377         for p=1:76
378             t(p)=p+64;
379             if (t(p) < tstar(4.*(i-1)+q,k))
380                 y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
381                     exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
382                         )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
383                             +q,k))))^(1/gamma(i))*m
384             else
385                 y(p)=m
386             end
387         end
388         subplot(2,2,3)
389         plot(t,y)
390         hold all
391         legend({'t^*=68','t^*=70','t^*=71'},'FontSize',8)
392         title('a=0.03')
393         xlabel('t')
394         ylabel('c^*(t)')
395         axis([65 80 0 inf])
396     end
397 end
398 end
399 for i=4:4 %gamma

```

```

398     for q=4:4 %a
399         for k=1:3 %columns of tstar
400             t=ones(76,1);
401             y=ones(76,1);
402             for p=1:76
403                 t(p)=p+64;
404                 if (t(p) < tstar(4.*(i-1)+q,k))
405                     y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
406                         exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
407                             )-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
408                             +q,k))))^(1/gamma(i))*m
409
410                     else
411                         y(p)=m
412                     end
413             end
414
415             subplot(2,2,4)
416             plot(t,y)
417             hold all
418             legend({'t^*=69','t^*=72','t^*=73'},'FontSize',8)
419             title('a=0.01')
420             xlabel('t')
421             ylabel('c^*(t)')
422             axis([65 80 0 inf])
423         end
424     end
425 end

```

```

422 annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal
    Consumption Function for \gamma =0.1', 'EdgeColor', 'none', '
    HorizontalAlignment', 'center', 'FontSize', 12)

```

2 MATLAB Coed for Accumulation of Wealth Function

```

1  j=0.03;
2  gamma=[3 1 0.5 0.1];
3  a=[0.10 0.05 0.03 0.01];
4  tstar=[73 80 84; 75 83 88; 77 85 89; 79 87 91; 70 74 76; 71 77
    80; 73 79 82; 75 81 84; 68 71 73; 70 73 76; 71 75 78; 73 78
    80; 66 67 68; 67 69 70; 68 70 71; 69 72 73];
5
6
7
8  figure
9  annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal
    Accumulated Wealth Function for \gamma =3', 'EdgeColor', 'none
    ', 'HorizontalAlignment', 'center', 'FontSize', 12)
10 for i=1:1 %gamma
11     for q=1:1 %a
12         for k=1:3
13             m=1;
14             fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).* tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.* tstar
                (4.*(i-1)+q,k))))-1)).*m.^(-gamma(i)))^(-1/gamma(i)

```



```

    )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
    ).*z)).^(1/gamma(i)))
15 S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
    -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
    -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
    exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
    -1)
16 S00=integral(S0,65,tstar(4*(i-1)+q,k))
17 t=ones(1,76);
18 S=ones(1,76);
19 I=ones(1,76);
20 y=ones(1,76);
21 for p=1:76
22     t(p)=p+64;
23     I(p)=integral(fun,65,t(p))
24     S(p)=exp(j*(t(p)-65)).*(S00*m+I(p))
25     for p=2:76
26         if ((S(p)<=0) || (S(p-1)==0) )
27             S(p)=0;
28         else
29             S(p)=S(p);
30         end
31     end
32 end
33 subplot(2,2,1)
34 plot(t(1:75),S(1:75))
35 hold all

```

```

36         legend({'t^*=73','t^*=80','t^*=84'},'FontSize',8)
37         title('\alpha = 0.10')
38         xlabel('t')
39         ylabel('S^*(t)')
40         axis([65 100 0 inf])
41     end
42 end
43 end
44
45 for i=1:1 %gamma
46     for q=2:2 %a
47         for k=1:3
48             m=1;
49             fun=@(z) (exp(-j.*(z-65)).*(1-(((exp((j-a(q)).*tstar
                    (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                    (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(1/gamma(i)
                    )).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                    ).*z)).^(1/gamma(i))))))
50             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                    -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                    -1)+q,k)))-1))).^(1/gamma(i))).*(exp(-0.00093.*(
                    exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
                    -1)
51             S00=integral(S0,65,tstar(4*(i-1)+q,k))
52             t=ones(1,76);
53             S=ones(1,76);
54             I=ones(1,76);

```

```

55         y=ones(1,76);
56         for p=1:76
57             t(p)=p+64;
58             I(p)=integral(fun,65,t(p))
59             S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
60             for p=2:76
61                 if ((S(p)<=0) || (S(p-1)==0) )
62                     S(p)=0;
63                 else
64                     S(p)=S(p);
65                 end
66             end
67         end
68         subplot(2,2,2)
69         plot(t(1:75),S(1:75))
70         hold all
71         legend({'t^*=75','t^*=83','t^*=88'},'FontSize',8)
72         title('\alpha = 0.05')
73         xlabel('t')
74         ylabel('S^*(t)')
75         axis([65 100 0 inf])
76     end
77 end
78 end
79
80 for i=1:1 %gamma
81     for q=3:3 %a

```

```

82     for k=1:3
83         m=1;
84         fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
            (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
            (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(-1/gamma(i)
            )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
            ).*z)).^(1/gamma(i))))
85         S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
            -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
            -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
            exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
            -1)
86         S00=integral(S0,65,tstar(4*(i-1)+q,k))
87         t=ones(1,76);
88         S=ones(1,76);
89         I=ones(1,76);
90         y=ones(1,76);
91         for p=1:76
92             t(p)=p+64;
93             I(p)=integral(fun,65,t(p))
94             S(p)=exp(j.*(t(p)-65)).*(S00*m+I(p))
95             for p=2:76
96                 if ((S(p)<=0) || (S(p-1)==0) )
97                     S(p)=0;
98                 else
99                     S(p)=S(p);
100             end

```

```

101         end
102     end
103     subplot(2,2,3)
104     plot(t(1:75),S(1:75))
105     hold all
106     legend({'t^*=77','t^*=85','t^*=89'},'FontSize',8)
107     title('\alpha = 0.03')
108     xlabel('t')
109     ylabel('S^*(t)')
110     axis([65 100 0 inf])
111 end
112 end
113 end
114
115 for i=1:1 %gamma
116     for q=4:4 %a
117         for k=1:3
118             m=1;
119             fun=@(z) (exp(-j.*(z-65)))*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))^(1/gamma(i)
                )).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                ).*z)).^(1/gamma(i))))
120             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                -1)+q,k)))-1))).^(1/gamma(i))).*(exp(-0.00093.*(
                exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i)))

```

```

-1)
121 S00=integral(S0,65,tstar(4*(i-1)+q,k))
122 t=ones(1,76);
123 S=ones(1,76);
124 I=ones(1,76);
125 y=ones(1,76);
126 for p=1:76
127     t(p)=p+64;
128     I(p)=integral(fun,65,t(p))
129     S(p)=exp(j*(t(p)-65))*(S00+m*I(p))
130     for p=2:76
131         if ((S(p)<=0) || (S(p-1)==0) )
132             S(p)=0;
133         else
134             S(p)=S(p);
135         end
136     end
137 end
138 subplot(2,2,4)
139 plot(t(1:75),S(1:75))
140 hold all
141 legend({'t^*=79','t^*=87','t^*=91'},'FontSize',8)
142 title('\alpha = 0.01')
143 xlabel('t')
144 ylabel('S^*(t)')
145 axis([65 100 0 inf])
146 end

```

```

147     end
148 end
149
150
151
152
153 figure
154 annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal
    Accumulated Wealth Function for \gamma =1', 'EdgeColor', 'none
    ', 'HorizontalAlignment', 'center', 'FontSize', 12)
155 for i=2:2 %gamma
156     for q=1:1 %a
157         for k=1:3
158             m=1;
159             fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(1/gamma(i)
                )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                ).*z)).^(1/gamma(i))))
160             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                -1)+q,k)))-1))).^(1/gamma(i))).*(exp(-0.00093.*(
                exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
                -1)
161             S00=integral(S0,65,tstar(4*(i-1)+q,k))
162             t=ones(1,76);
163             S=ones(1,76);

```

```

164         I=ones(1,76);
165         y=ones(1,76);
166         for p=1:76
167             t(p)=p+64;
168             I(p)=integral(fun,65,t(p))
169             S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
170             for p=2:76
171                 if ((S(p)<=0) || (S(p-1)==0) )
172                     S(p)=0;
173                 else
174                     S(p)=S(p);
175                 end
176             end
177         end
178         subplot(2,2,1)
179         plot(t(1:75),S(1:75))
180         hold all
181         legend({'t^*=70','t^*=74','t^*=76'},'FontSize',8)
182         title('\alpha = 0.10')
183         xlabel('t')
184         ylabel('S^*(t)')
185         axis([65 90 0 inf])
186     end
187 end
188 end
189
190 for i=2:2 %gamma

```



```

191     for q=2:2 %a
192         for k=1:3
193             m=1;
194             fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                    (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                    (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(-1/gamma(i)
                    )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                    ).*z)).^(1/gamma(i))))
195             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                    -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                    -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
                    exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
                    -1)
196             S00=integral(S0,65,tstar(4*(i-1)+q,k))
197             t=ones(1,76);
198             S=ones(1,76);
199             I=ones(1,76);
200             y=ones(1,76);
201             for p=1:76
202                 t(p)=p+64;
203                 I(p)=integral(fun,65,t(p))
204                 S(p)=exp(j.*(t(p)-65)).*(S00*m+I(p))
205                 for p=2:76
206                     if ((S(p)<=0) || (S(p-1)==0) )
207                         S(p)=0;
208                     else
209                         S(p)=S(p);

```

```

210         end
211     end
212 end
213 subplot(2,2,2)
214 plot(t(1:75),S(1:75))
215 hold all
216 legend({'t^*=71','t^*=77','t^*=80'},'FontSize',8)
217 title('\alpha = 0.05')
218 xlabel('t')
219 ylabel('S^*(t)')
220 axis([65 90 0 inf])
221 end
222 end
223 end
224
225 for i=2:2 %gamma
226     for q=3:3 %a
227         for k=1:3
228             m=1;
229             fun=@(z) (exp(-j.*(z-65)))*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(-1/gamma(i)
                )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                ).*z)).^(1/gamma(i))))
230             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(

```

```

        exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
        -1)
231 S00=integral(S0,65,tstar(4*(i-1)+q,k))
232 t=ones(1,76);
233 S=ones(1,76);
234 I=ones(1,76);
235 y=ones(1,76);
236 for p=1:76
237     t(p)=p+64;
238     I(p)=integral(fun,65,t(p))
239     S(p)=exp(j*(t(p)-65))*(S00+m*I(p))
240     for p=2:76
241         if ((S(p)<=0) || (S(p-1)==0) )
242             S(p)=0;
243         else
244             S(p)=S(p);
245         end
246     end
247 end
248 subplot(2,2,3)
249 plot(t(1:75),S(1:75))
250 hold all
251 legend({'t^*=73','t^*=79','t^*=82'},'FontSize',8)
252 title('\alpha = 0.03')
253 xlabel('t')
254 ylabel('S^*(t)')
255 axis([65 90 0 inf])

```

```

256         end
257     end
258 end
259
260 for i=2:2 %gamma
261     for q=4:4 %a
262         for k=1:3
263             m=1;
264             fun=@(z) (exp(-j.*(z-65)))*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(1/gamma(i)
                )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                ).*z)).^(1/gamma(i))))
265             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                -1)+q,k)))-1))).^(1/gamma(i))).*(exp(-0.00093.*(
                exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
                -1)
266             S00=integral(S0,65,tstar(4*(i-1)+q,k))
267             t=ones(1,76);
268             S=ones(1,76);
269             I=ones(1,76);
270             y=ones(1,76);
271             for p=1:76
272                 t(p)=p+64;
273                 I(p)=integral(fun,65,t(p))
274                 S(p)=exp(j.*(t(p)-65)).*(S00*m+I(p))

```

```

275         for p=2:76
276             if ((S(p)<=0) || (S(p-1)==0) )
277                 S(p)=0;
278             else
279                 S(p)=S(p);
280             end
281         end
282     end
283     subplot(2,2,4)
284     plot(t(1:75),S(1:75))
285     hold all
286     legend({'t^*=75','t^*=81','t^*=84'},'FontSize',8)
287     title('\alpha = 0.01')
288     xlabel('t')
289     ylabel('S^*(t)')
290     axis([65 90 0 inf])
291     end
292 end
293 end
294
295
296
297
298 figure
299 annotation('textbox',[0 0.9 1 0.1], 'String', 'Optimal
    Accumulated Wealth Function for \gamma =0.5', 'EdgeColor', '
    none', 'HorizontalAlignment', 'center', 'Fontsize',12)

```

```

300 for i=3:3 %gamma
301     for q=1:1 %a
302         for k=1:3
303             m=1;
304             fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                    (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                    (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(-1/gamma(i)
                    )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                    ).*z)).^(1/gamma(i))))
305             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                    -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                    -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
                    exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
                    -1)
306             S00=integral(S0,65,tstar(4*(i-1)+q,k))
307             t=ones(1,76);
308             S=ones(1,76);
309             I=ones(1,76);
310             y=ones(1,76);
311             for p=1:76
312                 t(p)=p+64;
313                 I(p)=integral(fun,65,t(p))
314                 S(p)=exp(j.*(t(p)-65)).*(S00*m+I(p))
315                 for p=2:76
316                     if ((S(p)<=0) || (S(p-1)==0) )
317                         S(p)=0;
318                     else

```

```

319             S(p)=S(p);
320         end
321     end
322 end
323 subplot(2,2,1)
324 plot(t(1:75),S(1:75))
325 hold all
326 legend({'t^*=68','t^*=71','t^*=73'},'FontSize',8)
327 title('\alpha = 0.10')
328 xlabel('t')
329 ylabel('S^*(t)')
330 axis([65 90 0 inf])
331 end
332 end
333 end
334
335 for i=3:3 %gamma
336     for q=2:2 %a
337         for k=1:3
338             m=1;
339             fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(1/gamma(i)
                )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                ).*z)).^(1/gamma(i))))
340             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i

```

```

-1)+q,k))) -1))).^( -1/gamma(i))) .* (exp(-0.00093.*(
exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
-1)
341 S00=integral(S0,65,tstar(4*(i-1)+q,k))
342 t=ones(1,76);
343 S=ones(1,76);
344 I=ones(1,76);
345 y=ones(1,76);
346 for p=1:76
347     t(p)=p+64;
348     I(p)=integral(fun,65,t(p))
349     S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
350     for p=2:76
351         if ((S(p)<=0) || (S(p-1)==0) )
352             S(p)=0;
353         else
354             S(p)=S(p);
355         end
356     end
357 end
358 subplot(2,2,2)
359 plot(t(1:75),S(1:75))
360 hold all
361 legend({'t^*=70','t^*=73','t^*=76'},'FontSize',8)
362 title('\alpha = 0.05')
363 xlabel('t')
364 ylabel('S^*(t)')

```



```

365         axis([65 90 0 inf])
366     end
367 end
368 end
369
370 for i=3:3 %gamma
371     for q=3:3 %a
372         for k=1:3
373             m=1;
374             fun=@(z) (exp(-j.*(z-65)))*(1-(((exp((j-a(q)).*tstar
                    (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                    (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(1/gamma(i)
                    )).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                    ).*z)).^(1/gamma(i))))
375             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                    -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                    -1)+q,k)))-1))).^(1/gamma(i))).*(exp(-0.00093.*(
                    exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
                    -1)
376             S00=integral(S0,65,tstar(4*(i-1)+q,k))
377             t=ones(1,76);
378             S=ones(1,76);
379             I=ones(1,76);
380             y=ones(1,76);
381             for p=1:76
382                 t(p)=p+64;
383                 I(p)=integral(fun,65,t(p))

```

```

384         S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
385     for p=2:76
386         if ((S(p)<=0) || (S(p-1)==0) )
387             S(p)=0;
388         else
389             S(p)=S(p);
390         end
391     end
392 end
393 subplot(2,2,3)
394 plot(t(1:75),S(1:75))
395 hold all
396 legend({'t^*=71','t^*=75','t^*=78'},'FontSize',8)
397 title('\alpha = 0.03')
398 xlabel('t')
399 ylabel('S^*(t)')
400 axis([65 90 0 inf])
401 end
402 end
403 end
404
405 for i=3:3 %gamma
406     for q=4:4 %a
407         for k=1:3
408             m=1;
409             fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                     (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar

```

```

(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)).^(-1/gamma(i)
)).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
).*z)).^(1/gamma(i)))
410 S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
-1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
-1)
411 S00=integral(S0,65,tstar(4*(i-1)+q,k))
412 t=ones(1,76);
413 S=ones(1,76);
414 I=ones(1,76);
415 y=ones(1,76);
416 for p=1:76
417     t(p)=p+64;
418     I(p)=integral(fun,65,t(p))
419     S(p)=exp(j*(t(p)-65)).*(S00*m+I(p))
420     for p=2:76
421         if ((S(p)<=0) || (S(p-1)==0) )
422             S(p)=0;
423         else
424             S(p)=S(p);
425         end
426     end
427 end
428 subplot(2,2,4)
429 plot(t(1:75),S(1:75))

```

```

430         hold all
431         legend({'t^*=73','t^*=78','t^*=80'},'FontSize',8)
432         title('\alpha = 0.01')
433         xlabel('t')
434         ylabel('S^*(t)')
435         axis([65 90 0 inf])
436     end
437 end
438 end
439
440
441
442
443 figure
444 annotation('textbox',[0 0.9 1 0.1],'String','Optimal
    Accumulated Wealth Function for \gamma =0.1','EdgeColor','
    none','HorizontalAlignment','center','Fontsize',12)
445 for i=4:4 %gamma
446     for q=1:1 %a
447         for k=1:3
448             m=1;
449             fun=@(z) (exp(-j.*(z-65)).*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i)
                )).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                ).*z)).^(1/gamma(i))))
450             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i

```

```

-1)+q,k)) .* exp(-0.00093.*(exp((0.087.* tstar(4.*(i
-1)+q,k)))-1)).^(-1/gamma(i)))*(exp(-0.00093.*(
exp(0.087.* x)-1)).*exp((j-a(q)).* x)).^(1/gamma(i))
-1)
451 S00=integral(S0,65,tstar(4*(i-1)+q,k))
452 t=ones(1,76);
453 S=ones(1,76);
454 I=ones(1,76);
455 y=ones(1,76);
456 for p=1:76
457     t(p)=p+64;
458     I(p)=integral(fun,65,t(p))
459     S(p)=exp(j*(t(p)-65))*(S00+m*I(p))
460     for p=2:76
461         if ((S(p)<=0) || (S(p-1)==0) )
462             S(p)=0;
463         else
464             S(p)=S(p);
465         end
466     end
467 end
468 subplot(2,2,1)
469 plot(t(1:75),S(1:75))
470 hold all
471 legend({'t^*=66','t^*=67','t^*=68'},'FontSize',8)
472 title('\alpha = 0.10')
473 xlabel('t')

```

```

474         ylabel('S^*(t)')
475         axis([65 80 0 inf])
476     end
477 end
478 end
479
480 for i=4:4 %gamma
481     for q=2:2 %a
482         for k=1:3
483             m=1;
484             fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(-1/gamma(i)
                )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                ).*z)).^(1/gamma(i))))
485             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
                exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
                -1)
486             S00=integral(S0,65,tstar(4*(i-1)+q,k))
487             t=ones(1,76);
488             S=ones(1,76);
489             I=ones(1,76);
490             y=ones(1,76);
491             for p=1:76
492                 t(p)=p+64;

```

```

493         I(p)=integral(fun,65,t(p))
494         S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
495         for p=2:76
496             if ((S(p)<=0) || (S(p-1)==0) )
497                 S(p)=0;
498             else
499                 S(p)=S(p);
500             end
501         end
502     end
503     subplot(2,2,2)
504     plot(t(1:75),S(1:75))
505     hold all
506     legend({'t^*=67','t^*=69','t^*=70'},'FontSize',8)
507     title('\alpha = 0.05')
508     xlabel('t')
509     ylabel('S^*(t)')
510     axis([65 80 0 inf])
511 end
512 end
513 end
514
515 for i=4:4 %gamma
516     for q=3:3 %a
517         for k=1:3
518             m=1;
519             fun=@(z) (exp(-j.*(z-65)).*(1-(((exp((j-a(q)).*tstar

```

```

(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(-1/gamma(i)
)).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
).*z)).^(1/gamma(i)))
520 S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
-1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
-1)
521 S00=integral(S0,65,tstar(4*(i-1)+q,k))
522 t=ones(1,76);
523 S=ones(1,76);
524 I=ones(1,76);
525 y=ones(1,76);
526 for p=1:76
527     t(p)=p+64;
528     I(p)=integral(fun,65,t(p))
529     S(p)=exp(j*(t(p)-65)).*(S00*m+I(p))
530     for p=2:76
531         if ((S(p)<=0) || (S(p-1)==0) )
532             S(p)=0;
533         else
534             S(p)=S(p);
535         end
536     end
537 end
538 subplot(2,2,3)

```



```

539         plot(t(1:75),S(1:75))
540         hold all
541         legend({'t^*=68','t^*=70','t^*=71'},'FontSize',8)
542         title('\alpha = 0.03')
543         xlabel('t')
544         ylabel('S^*(t)')
545         axis([65 80 0 inf])
546     end
547 end
548 end
549
550 for i=4:4 %gamma
551     for q=4:4 %a
552         for k=1:3
553             m=1;
554             fun=@(z) (exp(-j.*(z-65)).*(1-(((exp((j-a(q)).*tstar
                    (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                    (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i))).^(1/gamma(i)
                    )).*exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                    ).*z)).^(1/gamma(i))))
555             S0=@(x) exp(-j.*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                    -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                    -1)+q,k)))-1))).^(1/gamma(i))).*(exp(-0.00093.*(
                    exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
                    -1)
556             S00=integral(S0,65,tstar(4*(i-1)+q,k))
557             t=ones(1,76);

```

```

558     S=ones(1,76);
559     I=ones(1,76);
560     y=ones(1,76);
561     for p=1:76
562         t(p)=p+64;
563         I(p)=integral(fun,65,t(p))
564         S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
565         for p=2:76
566             if ((S(p)<=0) || (S(p-1)==0) )
567                 S(p)=0;
568             else
569                 S(p)=S(p);
570             end
571         end
572     end
573     subplot(2,2,4)
574     plot(t(1:75),S(1:75))
575     hold all
576     legend({'t^*=69','t^*=72','t^*=73'},'FontSize',8)
577     title('\alpha = 0.01')
578     xlabel('t')
579     ylabel('S^*(t)')
580     axis([65 80 0 inf])
581 end
582 end
583 end

```